Doc Dave

MPhys Notes

Advanced Dynamics

Dave Scott

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Chapter 1

Preliminaries

If I have seen further, it is by standing on the shoulders of giants.

— Isaac Newton, *Letter to Robert Hooke*
15th February 1676

1.1 Newton's Laws of Motion

In his *Philosophiae Naturalis Principia Mathematica*, Isaac Newton laid the foundations of classical mechanics by describing the relationships between bodies and forces acting upon them. He posited three laws of motion, which can be summarised as follows:

1. In an **inertial reference frame**, an object either remains at rest or continues to move at a constant velocity, unless acted upon by an external force, i.e. $\sum \mathbf{F} = 0$.

2. In an inertial reference frame the sum of the forces on an object is equal to the product of the mass of that object with its acceleration, i.e. $\mathbf{F} = m\mathbf{a}$.

3. When a body exerts a force on another, the second body simultaneously exerts an equal and opposite force on the first, i.e. $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

1.2 Tensor Analysis

In 1846, Irish physicist and mathematician William Hamilton introduced the term **tensor**, deriving from the modern Latin *tendere*, meaning 'to stretch'; however, since its introduction the term has changed definition in mathematical context. During the late 19th- and early 20th-Century a number of Italian, Dutch and German mathematicians contributed to the development of a compact notation for dealing with tensors, known as tensor analysis.

Definition 1.1: Tensor

In mathematics, a **tensor** is an object which defines multidimensional arrays. Tensors have **ranks** which denote the number of dimensions in which the tensor acts or can be written, i.e. rank-0 are scalars, rank-1 are vectors, rank-2 are matrices. The notation for tensors uses sub- and superscripts to denote components such that each set of values identifies an element of the array, i.e. row vector $\mathbf{a} = \begin{pmatrix} a_1 & a_2 & \dots & a_N \end{pmatrix}$.

Classically the three-dimensional position vector of an object $\mathbf{r}$ is written in Cartesian coordinates as $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. This can be rewritten in terms of **orthogonal** unit vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z = \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$, now giving the position vector as $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$. However, tensors are not necessarily limited to a finite number of dimensions and so denoting tensor components with letters is not a practical solution. To circumvent this problem, the Italian mathematician Tullio Levi-Civita introduced numeric indexing c. 1900, i.e. $x, y, z \to x_1, x_2, x_3; \mathbf{e}_{x,y,z} \to \mathbf{e}_{1,2,3}$, hence $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \sum_{i=1}^{3} x_i\mathbf{e}_i$.

Definition 1.2: Orthogonal

Tensors are **orthogonal** if their inner product is equal to zero, i.e. two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$. Geometrically this can be interpreted as the vectors being at right angles to each other.

In 1916, Albert Einstein introduced a new notation - Einstein summation notation - such that the summation over the index is implied, giving the final form of the position vector as $\mathbf{r} = x_i\mathbf{e}_i$. In general, the magnitudes x_i and directions $\mathbf{e}_i$ of the vector are functions of time, i.e. $x_i = x_i(t); \mathbf{e}_i = \mathbf{e}_i(t)$.

1.2.1 The Kronecker Delta

The **Kronecker delta**, denoted δ_{ij}, was introduced by German mathematician Leopold Kronecker in 1866 and is defined by

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j, \end{cases}$$

which can be represented by the **identity matrix**.

Definition 1.3: Identity Matrix

The **identity matrix**, denoted $\mathbb{I}_n$ or $\mathbb{1}_n$, is an $n \times n$ matrix with ones on the main diagonal and zeros elsewhere, i.e.

$$\mathbb{I}_1 = [1], \mathbb{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbb{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ldots$$

When acting upon an $m \times n$ matrix A, the identity matrix leaves the original matrix unchanged, i.e.

$$\mathbb{I}A = A\mathbb{I} = A.$$

The inner products of two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \ldots + a_N b_N,$$

however the sum of all permutations of component products is given by

$$\sum_{i=1}^{N} \sum_{j=1}^{N} a_i b_j = \begin{array}{c} a_1 b_1 + a_1 b_2 + \ldots + a_1 b_N \\ a_2 b_1 + a_2 b_2 + \ldots + a_2 b_N \\ \vdots \\ a_N b_1 + a_N b_2 + \ldots + a_N b_N. \end{array}$$

It can be seen that the inner product of two matrices is then given by the components of this sum for which the indices are equal, thus can be written in terms of the Kronecker delta as

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i \delta_{ij} b_j,$$

which in summation notation becomes

$$\mathbf{a} \cdot \mathbf{b} = a_i \delta_{ij} b_j.$$

1.2.2 The Levi-Civita Symbol

Tullio Levi-Civita introduced his namesake symbol ϵ_{ijk} to tensor analysis in his 1927 magnum opus *The Absolute Differential Calculus*. The definition of the **Levi-Civita symbol** (or **permutation symbol**) is

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } i, j, k \text{ is an even permutation of 1, 2, 3} \\ 0 & \text{any two indices are equal} \\ -1 & \text{if } i, j, k \text{ is an odd permutation of 1, 2, 3.} \end{cases}$$

Even permutations are those for which the ordering of ijk is either 123, 231 or 312. Conversely, odd permutations are those for which the ordering of ijk is either 321, 213 or 132. This might be better understood by writing the indices as a cycle and taking note of the direction in which the indices rotate (if at all), as shown in Figure 1.1.

Figure 1.1: Left: Clockwise rotation of indices gives $\epsilon_{ijk} = +1$. Right: Anticlockwise rotation of indices gives $\epsilon_{ijk} = -1$.

In terms of its indices, the Levi-Civita symbol can be written

$$\epsilon_{ijk} = \frac{1}{2} (i - j) (j - k) (k - i),$$

which satisfies the definition. For example, the index ordering 312 has $i = 3$, $j = 1$ and $k = 2$, and so the Levi-Civita is $\epsilon_{312} = \frac{1}{2}(3-1)(1-2)(2-3) = 2$, which was expected as it is an even index permutation.

The main application of the Levi-Civita epsilon to tensor analysis is in taking matrix determinants. The determinant of a 3×3 matrix A is given by

$$\det A = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \begin{array}{r} A_{11}\left(A_{22}A_{33} - A_{23}A_{32}\right) \\ -A_{12}\left(A_{21}A_{33} - A_{23}A_{31}\right) \\ +A_{13}\left(A_{21}A_{32} - A_{22}A_{31}\right) \end{array} = \begin{array}{r} A_{11}A_{22}A_{33} - A_{11}A_{23}A_{32} \\ +A_{12}A_{23}A_{31} - A_{12}A_{21}A_{33} \\ +A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31}. \end{array}$$

Considering all possible permutations (as in Subsection 1.2.1), one sees that these are the terms for which the second index for each component set are never equal. Checking whether the term is added or subtracted, one sees that the determinant of a matrix can be written as $\det A = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} A_{1i} A_{2j} A_{3k}$. When applied to the cross product of two vectors $\mathbf{a}$ and $\mathbf{b}$, this becomes

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{array}{r} \mathbf{e}_1 a_2 b_3 - \mathbf{e}_1 a_3 b_2 \\ +\mathbf{e}_2 a_3 b_1 - \mathbf{e}_2 a_1 b_3 \\ +\mathbf{e}_3 a_1 b_2 - \mathbf{e}_3 a_2 b_1 \end{array} = \sum_{i,j,k=1}^{3} \epsilon_{ijk} \mathbf{e}_i a_j b_k,$$

which in summation notation is

$$\mathbf{a} \times \mathbf{b} = \epsilon_{ijk} \mathbf{e}_i a_j b_k.$$

Using this notation it can be shown that the cross product of a vector with itself is zero in all components.

Proof 1.1: a × a = 0

In summation notation the cross product of a vector $\mathbf{a}$ with itself is

$$\mathbf{a}\times\mathbf{a}=\epsilon_{ijk}\mathbf{e}_i a_j a_k.$$

This can be expanded as

$$\mathbf{a}\times\mathbf{a}=\begin{matrix}\mathbf{e}_1\left(a_2a_3-a_3a_2\right)\\+\mathbf{e}_2\left(a_3a_1-a_1a_3\right)\\+\mathbf{e}_3\left(a_1a_2-a_2a_1\right)\end{matrix}=\begin{matrix}\mathbf{e}_1 0\\+\mathbf{e}_2 0\\+\mathbf{e}_3 0\end{matrix}=\mathbf{0},$$

where $\mathbf{0}$ is known as the **zero vector**, the vector for which all components are zero.

1.3 Linear Dynamics

Linear dynamics considers only motion in straight lines and hence ignores any angular changes that result from time derivatives. Prior to discussing linear dynamical properties, basic mathematical definitions to kinematic quantities will be briefly presented:

Position vector	$\mathbf{r}=x\mathbf{e}_x+y\mathbf{e}_y+z\mathbf{e}_z=\sum_{i=1}^{3}x_i\mathbf{e}_i\equiv x_i\mathbf{e}_i$
Velocity	$\mathbf{v}:=\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t)=\dot{\mathbf{r}}(t)$
Speed	$v:=\lvert\mathbf{v}\rvert=\sqrt{\mathbf{v}\cdot\mathbf{v}}=\sqrt{v_x^2+v_y^2+v_z^2}$
Acceleration	$\mathbf{a}(t):=\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}(t)\ddot{\mathbf{r}}(t)=\dot{\mathbf{v}}(t).$

1.3.1 Momentum

Classically the momentum of an object $\mathbf{p}$ is given by $\mathbf{p}=m\mathbf{v}$. This definition assumes that the momentum is constant with time, hence a more generalised definition considers any possible variations with time, i.e.

$$\mathbf{p}(t):=m(t)\mathbf{v}(t).$$

1.3.2 Force

Force is defined as the rate of change of momentum, i.e.

$$\mathbf{F} := \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\,.$$

Using the defintion of momentum from Section 1.3.1 and the product rule, the force can be shown to be given by

$$\mathbf{F} = \frac{\mathrm{d}m}{\mathrm{d}t}\,\mathbf{v} + m\,\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\,.$$

Newton's second law of motion, as stated in Section 1.1, is hence a special case of this equation, for which mass is conserved, i.e. $\frac{\mathrm{d}m}{\mathrm{d}t} = 0$. The more generalised form introduced here includes the rate of change of mass, which has such applications as fuel usage in rockets and cars.

1.4 Angular Dynamics

1.4.1 Momentum

Angular momentum $\mathbf{L}$ is defined as the cross product of the position with the linear momentum, i.e.

$$\mathbf{L} := \mathbf{r} \times \mathbf{p}.$$

1.4.2 Torque

Analoguous to force being the time derivative of linear momentum, torque is the time derivative of angular momentum, i.e.

$$\mathbf{M} := \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{r} \times \mathbf{F}.$$

Proof 1.2: $\mathbf{M} = \mathbf{r} \times \mathbf{F}$

From the definition of angular momentum, one gets the torque as

$$\mathbf{M} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{r} \times \mathbf{p}\right).$$

Using the product rule and the definitions of momentum and linear force, this becomes

$$\begin{aligned}\mathbf{M} &= \tfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{p} + \mathbf{r} \times \tfrac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \\ &= m \underbrace{\mathbf{v} \times \mathbf{v}}_{=0} + \mathbf{r} \times \mathbf{F}.\end{aligned}$$

1.5 Two-Body Systems

1.5.1 The Two-Body System

Consider two point-like bodies of masses m_1 and m_2, where $m_1 < m_2$, interacting via an attractive force such as gravitation or electromagnetism, as shown in Figure 1.2. The force on body 2 due to body 1 $\mathbf{F}$ is given by

$$\mathbf{F} = \mathbf{F}_{21} = -\mathbf{F}_{12}.$$

The total mass of a multibody system M is given by the sum of all masses within the system, i.e. for the two-body system described above

$$M = m_1 + m_2.$$

The centre-of-mass of a system $\mathbf{R}$ is the point which represents the mean position of all matter in the system, i.e. for this two-body system

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}.$$

The relative position of the first body to the second body $\mathbf{r}$ is given by

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2.$$

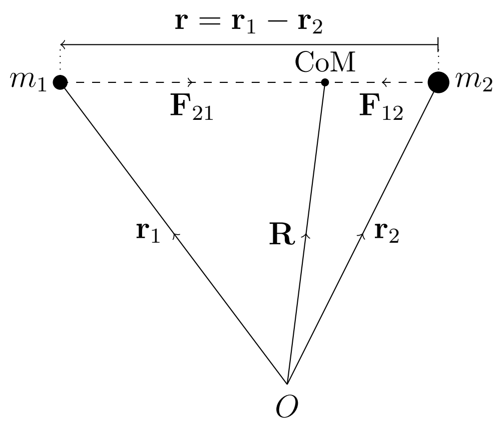

Figure 1.2: Two bodies of respective m_1 and m_2 are located at $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. The centre-of-mass of the two-body system is located at $\mathbf{R}$.

The definitions given above imply the following relations:

$$\begin{aligned} M\mathbf{R} &= m_1\mathbf{r}_1 + m_2\mathbf{r}_2 \\ m_1\mathbf{r} &= m_1\mathbf{r}_1 - m_1\mathbf{r}_2 \\ m_2\mathbf{r} &= m_2\mathbf{r}_1 - m_2\mathbf{r}_2. \end{aligned}$$

From these, one can obtain $\mathbf{r}_{1,2}$ in terms of the other variables, i.e.

$$\begin{aligned} M\mathbf{R} + m_2\mathbf{r} &= m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + (m_2\mathbf{r}_1 - m_2\mathbf{r}_2) \\ &= \underbrace{(m_1 + m_2)}_{=M}\mathbf{r}_1 \\ \Rightarrow \mathbf{r}_1 &= \mathbf{R} + \frac{m_2}{M}\mathbf{r}. \end{aligned}$$

$$\begin{aligned} M\mathbf{R} - m_1\mathbf{r} &= m_1\mathbf{r}_1 + m_2\mathbf{r}_2 - (m_1\mathbf{r}_1 - m_1\mathbf{r}_2) \\ &= \underbrace{(m_1 + m_2)}_{=M}\mathbf{r}_2 \\ \Rightarrow \mathbf{r}_2 &= \mathbf{R} - \frac{m_1}{M}\mathbf{r}; \end{aligned}$$

Defining the velocity of the centre-of-mass as $\mathbf{V} \coloneqq \dot{\mathbf{R}}$ and the relative velocity (rate at which the particles move relative to one another) $\mathbf{v} \coloneqq \dot{\mathbf{r}}$, the individual particle velocties are then given by

$$\mathbf{v}_1 = \dot{\mathbf{R}} + \frac{m_2}{M}\dot{\mathbf{r}}_1 \qquad \mathbf{v}_2 = \dot{\mathbf{R}} - \frac{m_1}{M}\dot{\mathbf{r}}_2$$
$$= \mathbf{V} + \frac{m_2}{M}\mathbf{v}_1 \qquad = \mathbf{V} - \frac{m_1}{M}\mathbf{v}_2.$$

1.5.2 Equations of Motion

From Newton's 2nd and 3rd laws, the equations of motion for the two bodies are

$$\begin{aligned} &\text{Body 1} \quad \mathbf{F}_{21} = m_1\ddot{\mathbf{r}}_1 = \mathbf{F} \\ &\text{Body 2} \quad \mathbf{F}_{12} = m_2\ddot{\mathbf{r}}_2 = -\mathbf{F}. \end{aligned}$$

The sum of these is

$$\mathbf{F}_{21} + \mathbf{F}_{12} = 0,$$

which is

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = M \underbrace{\frac{m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2}{M}}_{=\ddot{\mathbf{R}}} = 0.$$

This implies that $\dot{\mathbf{R}} = \mathbf{V} =$ constant, thus the centre of mass is an inertial frame, i.e. Newton's 1st law.

The equation of motion of the two-body system is then

$$\begin{aligned} \mathbf{F} = m_1\ddot{\mathbf{r}}_1 &= m_1\left(\underbrace{\ddot{\mathbf{R}}}_{=0} + \frac{m_2}{M}\ddot{\mathbf{r}}\right) \\ &= \frac{m_1 m_2}{m_1 + m_2}\ddot{\mathbf{r}}. \end{aligned}$$

The mass term is called the **reduced mass**, denoted $\mu \coloneqq \frac{m_1 m_2}{m_1+m_2}$, thus the equation of motion of a two-body system is

$$\mathbf{F} = \mu\ddot{\mathbf{r}}.$$

1.5.3 Total Angular Momentum

The angular momentum is given by the sum of the individual angular momenta, where the angular momentum for each body is given by $\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v}$, thus the angular momentum of a two-body system is

$$\mathbf{L} = M\mathbf{R} \times \mathbf{V} + \mu\mathbf{r} \times \mathbf{v},$$

where the first term is the angular momentum of the centre-of-mass frame, whereas the second term is that of the relative system.

Proof 1.3: $\mathbf{L} = M\mathbf{R} \times \mathbf{V} + \mu\mathbf{r} \times \mathbf{v}$

$$\begin{aligned}
\mathbf{L} &= \mathbf{L}_1 + \mathbf{L}_2 \\
&= m_1\mathbf{r}_1 \times \mathbf{v}_1 + m_2\mathbf{r}_2 \times \mathbf{v}_2 \\
&= m_1\left(\mathbf{R} + \frac{m_2}{M}\mathbf{r}\right) \times \left(\mathbf{V} + \frac{m_2}{M}\mathbf{v}\right) + m_2\left(\mathbf{R} - \frac{m_1}{M}\mathbf{r}\right) \times \left(\mathbf{V} - \frac{m_1}{M}\mathbf{v}\right) \\
&= \mathbf{R} \times \mathbf{V}\underbrace{(m_1 + m_2)}_{=M} + \mathbf{R} \times \mathbf{v}\underbrace{\left(\frac{m_1 m_2}{M} - \frac{m_2 m_1}{M}\right)}_{=0} \\
&\qquad + \mathbf{r} \times \mathbf{V}\underbrace{\left(\frac{m_1 m_2}{M} - \frac{m_2 m_1}{M}\right)}_{=0} + \mathbf{r} \times \mathbf{v}\underbrace{\left(\frac{m_1 m_2^2 + m_2 m_1^2}{M^2}\right)}_{=\mu} \\
&= M\mathbf{R} \times \mathbf{V} + \mu\mathbf{r} \times \mathbf{v}.
\end{aligned}$$

1.5.4 Total Kinetic Energy

The total kinetic energy of the two-body system T is given by

$$T = \frac{1}{2}M\mathbf{V}^2 + \frac{1}{2}\mu\mathbf{v}^2.$$

where the first term is the kinetic energy of the centre-of-mass frame and the second term is the kinetic energy of the relative system.

Proof 1.4: $T = \frac{1}{2}M\mathbf{V}^2 + \frac{1}{2}\mu\mathbf{v}^2$

The total kinetic energy of a system is given by the sum of the individual kinetic energies, i.e. for a two-body system

$$\begin{aligned}
T &= T_1 + T_2 \\
&= \frac{1}{2}m_1\mathbf{v}_1^2 + \frac{1}{2}m_2\mathbf{v}_2^2 \\
&= \frac{1}{2}m_1\left(\mathbf{V} + \frac{m_2}{M}\mathbf{v}\right)^2 + \frac{1}{2}m_2\left(\mathbf{V} - \frac{m_1}{M}\mathbf{v}\right)^2 \\
&= \frac{1}{2}m_1\left(\mathbf{V}^2 + 2\frac{m_2}{M}\mathbf{V}\cdot\mathbf{v} + \frac{m_2^2}{M^2}\mathbf{v}^2\right) \\
&\quad\quad + \frac{1}{2}m_2\left(\mathbf{V}^2 - 2\frac{m_1}{M}\mathbf{V}\cdot\mathbf{v} + \frac{m_1^2}{M^2}\mathbf{v}^2\right) \\
&= \frac{1}{2}\left(m_1 + m_2\right)\mathbf{V}^2 + \frac{1}{2}\frac{m_1m_2^2 + m_2m_1^2}{M^2}\mathbf{v}^2 \\
&= \frac{1}{2}M\mathbf{V}^2 + \frac{1}{2}\mu\mathbf{v}^2.
\end{aligned}$$

Chapter 2

Gravitation

2.1 Force Fields and Potentials

Definition 2.1: Force field

A **force field**, denoted $\mathbf{F}(\mathbf{r})$, is a vector which describes a force whose magnitude and direction are position-dependent.

A **conservative** force field is one which is derived from a potential $U(\mathbf{r})$, i.e.

$$\begin{aligned}\mathbf{F}(\mathbf{r}) &= -\boldsymbol{\nabla} U(\mathbf{r}) \\ &= -\left(\mathbf{e}_x \frac{\partial U}{\partial x} + \mathbf{e}_y \frac{\partial U}{\partial y} + \mathbf{e}_z \frac{\partial U}{\partial z}\right) \\ &= -\left(\mathbf{e}_1 \frac{\partial U}{\partial x_1} + \mathbf{e}_2 \frac{\partial U}{\partial x_2} + \mathbf{e}_3 \frac{\partial U}{\partial x_3}\right) \\ &= -\mathbf{e}_i \frac{\partial}{\partial x_i} U(\mathbf{r}),\end{aligned}$$

where $\boldsymbol{\nabla} \equiv \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z} = \mathbf{e}_i \frac{\partial}{\partial x_i}$ is called **del** or **nabla**. The time derivative of this potential is given by the chain rule as

$$\frac{\mathrm{d}U(\mathbf{r}(t))}{\mathrm{d}t} = \dot{x} \frac{\partial U}{\partial x} + \dot{y} \frac{\partial U}{\partial y} + \dot{z} \frac{\partial U}{\partial z} = \dot{\mathbf{r}} \cdot \boldsymbol{\nabla} U = \mathbf{v} \cdot \boldsymbol{\nabla} U.$$

The work done within a conservative force field is independent of path.

Proof 2.1: Work done within a conservative field is independent of path

The work done in moving a body is defined as the integral of the force applied $\mathbf{F}$ over the path taken C, i.e.

$$W := \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

As a result, the work done moving a body between two points $\mathbf{r}_A$ and $\mathbf{r}_B$ within a conservative force field is given by

$$\begin{aligned}
W_{AB} &= \int_{\mathbf{r}_A}^{\mathbf{r}_B} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\
&= \int_{\mathbf{r}_A}^{\mathbf{r}_B} (-\boldsymbol{\nabla} U) \cdot \mathrm{d}\mathbf{r} \\
&= \int_{t_A}^{t_B} (-\boldsymbol{\nabla} U) \cdot \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right) \mathrm{d}t \\
&= -\int_{t_A}^{t_B} \frac{\partial U}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial U}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial U}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t \\
&= -\int_{t_A}^{t_B} \frac{\mathrm{d}U}{\mathrm{d}t}\,\mathrm{d}t \\
&= U(\mathbf{r}(t_A)) - U(\mathbf{r}(t_B)) \\
&= U(\mathbf{r}_A) - U(\mathbf{r}_B).
\end{aligned}$$

As W is dependent on the potentials at only the start- and endpoints, the work done is independent of the path taken.

A **central** force field is a force field for which the force is only dependent on the distance from a fixed point O, which results in all forces being either inward or outward from the origin. Writing the force $\mathbf{F}(\mathbf{r})$ as the product of a scalar force $F(r)$ and a radial unit vector $\hat{\mathbf{r}}$, one sees that $F(r) = \text{const}$ on a sphere ($r = \text{const}$), i.e.

$$\mathbf{F}(\mathbf{r}) = F(r)\hat{\mathbf{r}} = F(r)\frac{\mathbf{r}}{r}.$$

This central force field behaviour can be seen in Newtonian gravity, in which the force is not angularly dependent and varies only with the radial distance

r and masses of the bodies m_1 and m_2, i.e.

$$\mathbf{F}_G = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}},$$

where the negative sign denotes an attractive force, $G = 6.67\times10^{-11}$ m^3 kg^{-1} s^{-2} is Newton's gravitational constant and $\hat{\mathbf{r}}$ denotes the directionality of the central force field.

In order to obtain the potential for a central force field, one must apply an inverse operation to the definition such that the gradient operator is countered. The operation to do such is a line integral, i.e.

$$\int_{r_0}^{r} \boldsymbol{\nabla} U \cdot \mathrm{d}\boldsymbol{\ell} = U(r) - U(r_0).$$

The potential of a central field is hence given in terms of the scalar force as

$$U(r) = -\int_{r_0}^{r} F(r')\hat{\mathbf{r}}' \cdot \mathrm{d}\boldsymbol{\ell} + U(r_0) = -\int_{r_0}^{r} F(r')\ \mathrm{d}r' + U(r_0).$$

The lower bound r_0 is often chosen such that the constant $U(r_0) = 0$, e.g. the gravitational potential energy of an object on the ground is zero but when raised above the ground is non-zero. One can check that the above definition is self-consistent, i.e. it

2.2 Kepler's Motion in a Central Force Field

Following observations, in 16..... the Danish astronomer and physicist Johannes Kepler formalised three laws of motion of celestial objects. His laws can be summarised as follows:

2.3 Particle Orbits as Conic Sections

2.4 Kepler's Laws

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