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PhD Notes

Beam Dynamics

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Chapter 1

Introduction to Beam Dynamics

1.1 Introduction

1.1.1 The Synchrotron

The pathway to becoming an accelerator physicst involves learning special relativity, electromagnetism and classical mechanics. These can then be applied to create an understanding of accelerator beam dynamics, which can itself can be split into transverse and longitudinal beam dynamics.

1.1.2 Periodicity and Stability

Consider an idealised pendulum system comprised of a mass on the end of a string. We see periodic oscillations which go on forever, so the motion is stable. The period of the motion depends on gravity and the length of the pendulum string. This oscillator system can be used to describe the motion of a beam in an accelerator.

1.1.3 Hierarchy of Beam Descriptions

We do beam dynamics to understand the motion of particles in linear and circular accelerators in order to

- Understand the fundamentals of existing machines
- Optimise and commission accelerators
- Design new machines, e.g. a new collider
- Design novel machines, e.g. a non-scaling FFAG

The fundamental tool of a person 'doing' beam dynamics is knowing how to calculate the motion of a charged particle in a real electromagnetic field;

- Magnetostatic configurations
- Time-dependent fields
- 'Optics'
- The approximations used
- How the particles interact with their surroundings
- How the particles in bunches interact with one another

The most basic question is how to represent the beam passing through an accelerator in beam dynamics language, which leads to the hierarchy of beam descriptions.

Consider a monatomic gas in a box. The state of this system can be described by several numbers;

- Pressure, P
- Temperature, T
- Volume, V
- Number of moles, n.

An equation of state relates these quantities together. An ideal gas obeys the ideal gas law

$$PV = nRT,$$

which relates the state variables to each other and descirbes how they change, e.g. an isobaric change of state maintains constancy of pressure.

If, instead, the gas is also made up of gas molecules, each with a position and a momentum in every degree of freedom of the system. Each molecule has a speed and a kinetic (translational) energy. This is a **microscopic view** of a gas in a box, and it is an equally valid view to the monatomic gas picture. The average kinetic energy, K_{tr}, is directly proportional to the temperature of the gas;

$$K_{tr} = \frac{3}{2}nRT.$$

It is common in physics to have several different, but equivalent, views of the same physical system;

- Physics of an ideal gas
- Quantum mechanics, with wave and matrix formulations.

The same situation happens in beam dynamics;

- **Global view**
  We assume a ring or beamline exists and study the global properties of the system, e.g. stability or tune.
- **Local view**
  We consider the details of the machine, e.g. which the best frame of reference is, how the magnets appear in this frame, how to combine the frame of many magnets, how a single particle moves in the system etc.

1.1.4 Newton and Hamilton

As with beam dynamics, there are several equivalent ways of 'doing' particle dynamics, each of which can be used to solve dynamical problems. The three formulations of dynamics are:

1. Newtonian dynamics
2. Lagrangian dynamics

3. Hamiltonian dynamics.

The formulation choice depends on the kind of problem being attempted. In accelerator physics we tend to use Newtonian and Hamiltonian dynamics, and each one has its own merits.

Newtonian Dynamics

Newton's formulation involves relating a force to the resulting motion via his second law, namely

$$\frac{d}{dt} m\dot{\mathbf{x}} = F(\dot{\mathbf{x}}, \mathbf{x}, t).$$

Doing the physics means working very hard to calculate the force, F, which depends on the particle positions, velocities and on the independent variable, time t. We then solve Newton's second law to calculate the evolution of the position and velocity of a particle as a function of time.

Consider a mass, m, on a spring with spring constant k. The force on the mass from the spring pushes the mass back to the equilibrium point is given by

$$F = -kx.$$

Using Newton's laws gives the equation of motion as

$$\begin{aligned} m\frac{d}{dt}\dot{x} &= -kx \\ \Rightarrow \frac{d^2x}{dt^2} &= -\frac{k}{m}x. \end{aligned}$$

This can instead be written as

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

with the angular frequency, ω, being given by

$$\omega = \sqrt{\frac{k}{m}}.$$

This standard equation has an oscillatory solution with two constants;

$$x(t) = x_0 \sin(\omega t + \phi_0),$$

which can be written as

$$x(t) = c_1 \sin(\omega t) + c_2 \cos(\omega t).$$

For the same oscillator with the opposite force, i.e. $F = +kx$, the standard equation of motion becomes

$$\frac{d^2x}{dt^2} = \frac{k}{m}x.$$

The force is now pushing the particle to larger amplitudes, hence the solution is hyperbolic;

$$x(t) = c_1 \cosh(\omega t) + c_2 \sinh(\omega t).$$

A very important equation for accelerator physics is the Lorentz force law, which relates the force experienced by a particle of charge q from electric and magnetic fields;

$$\mathbf{F} = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}).$$

The vector nature of this equation is very important. $\mathbf{E}$ and $\mathbf{B}$ themselves obey equations, which specify their vector fields in the presence of sources and boundary conditions. Usually the speed is close to the speed of light and so magnetic fields are efficient to guide particles.

Hamiltonian Dynamics

Hamiltonain dynamics is the formulation of mechanics by William R. Hamilton in 1833. This way of working carries the advantage of a well-defined mathematical framework, where keeping control of variables, invariants and approximations is much easier than when using Newtonian dynamics. Of central importance is the Hamiltonain function, H, which is a part of a set of conjugate variable pairs. For example, in one dimension we would deal with the position and the canonical momenta of that particle as a conjugate pair.

Given a Hmailtonian which is a function of the canonical variables, the dynamics are worked out using Hamilton's equations:

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}$$
$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}.$$

For a system with n dimensions, this gives a set of $2n$ first order ODEs to solve. Conceptually the Hamiltonian plays the role that the force does in Newtonian dynamics. Conventionally, students of accelerator physics learn the way with Newton first and way the way of Hamilton second. In this course we will largely adopt this conventional approach, of using forces first.

Example 1.1: Harmonic Oscillator

The Hamiltonian for a one-dimensional harmonic oscillator is given by

$$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

in terms of the particle position and canonical momenta. Hamilton's equations directly give

$$\frac{dx}{dt} = \frac{p_x}{m}$$
$$\frac{dp_x}{dt} = -m\omega^2 x,$$

which are equivalent to the equations of motion obtained from Newton's formulation of dynamics. Note that the Hamiltonian can be written as the sum of the kinetic and potential energies;

$$H = T + V.$$

The total time derivative of the Hamiltonian depends on x and p_x:

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial p_x}\frac{dp_x}{dt} + \frac{\partial H}{\partial t}.$$

Hamilton's equations can be used to rewrite some of the derivatives:

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial H}{\partial x} + \frac{\partial H}{\partial t}.$$

Therefore if the Hamiltonian does not explicitly depend on time then

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0,$$

and so H is a conserved quantity of the system.

1.2 Hill's Equations

So far we have seen that the equation for simple harmonic oscillation, with different signs for the spring constant, can lead to oscillating and diverging solutions. The basic equations of motion for a particle in an accelerator are called Hill's equation, which will soon be derived to be

$$x'' + \left(k + \frac{1}{\rho^2}\right) x = 0$$
$$z'' - kz = 0.$$

The second equation is the equation of motion for a region with no bending. Depending on the sign of the constant this corresponds to either sine/cosine or sinh/cosh solutions for $z(s)$. It therefore seems that an accelerator, at least linearly, may be thought of as being made up of piece-wise regions with different sizes and signs of the spring constant;

$$x''(s) + \left(k(s) + \frac{1}{\rho(s)^2}\right) x(s) = 0$$

1.2.1 Beams and Magnets

Put simply, an accelerator uses electric and magnetic fields to accelerate and control particles. The force, $\mathbf{F}$, on a particle with charge q moving at velocity $\dot{\mathbf{v}}$ in electrical and magnetic fields is given by

$$\mathbf{F} = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}).$$

Considering a magnetic field of strength $\mathbf{B} = 1$ T, the force exerted on a particle travelling at the speed of light is given by

$$F_B = q \times 3 \times 10^8 \ \mathrm{m\,s^{-1}} \times 1 \ \mathrm{V\,m^{-2}} = q \times 300 \ \mathrm{MV\,m^{-1}}.$$

Given that making electric fields around 1 $\mathrm{MV\,m^{-1}}$ is difficult, there is a great benefit in using magnetic fields to deflect the beams. However, electric fields will still be using to accelerate the particles as magnetic fields can do no work and so cannot increase the particles' energies.

We wish to make the particle beam move in a circle, so we apply a constant magnetic field. The force on the particle is always perpendicular to the motion. The magnetic bending force is given by

$$F = qvB,$$

which must be equal to the centripetal force which is given by

$$F = \frac{\gamma m v^2}{\rho},$$

where ρ is the radius of the orbit. Therefore

$$qB = \frac{\gamma m v}{\rho},$$

which is often written as the **beam rigidity**, $B\rho$;

$$B\rho = \frac{\gamma m v}{q} = \frac{p}{q}.$$

Therefore if a particle of charge q with momentum p traverses a guide field B, it will follow a circular path of radius ρ. Alternatively, the product of the field and the radius is a function of only the particle's charge and momentum. The particular combination $\frac{p}{q}$ is exact;y that which appears in the normalisation of many physical quantities in beam dynamics. Note that the beam rigidity does not refer to any specific field strength or radius of curvature of the trajectory. It should be thought of as simply another way of writing the reference momentum.

For the usual case of the beam being comprised of single-charge particles, i.e. $q = \pm e$, with total energy much greater than the rest energy, i.e. $E_0 \gg mc^2$, then the beam rigidity can be calculated from

$$B\rho \ \mathrm{T\,m} = \frac{E_0}{c}\frac{\mathrm{eV}}{\mathrm{m\,s}}.$$

The magnetic field strengths can be normalised to the beam momenta, allowing energy-independent "k-values" to be obtained;

$$k_0 = \frac{q}{p}B.$$

Furthermore, magnets can provide multiple fields, e.g. a dipole and quadrupole field, known as a **combined function magnet**.

It makes sense to take some arbitrary guide field and make a Taylor expansion about the design orbit and normalise to the beam rigidity;

$$B_z(x) = B_{z0} + \frac{dB_z}{dx}x + \frac{1}{2}\frac{d^2B_z}{dx^2}x^2 + \frac{1}{3!}\frac{d^3B_z}{dx^3}x^3 + \ldots.$$

Therefore, by multiplying both sides by the charge to momentum ratio, $\frac{e}{p}$;

$$\begin{aligned}\frac{e}{p}B_z(x) &= \frac{e}{p}B_{z0} + \frac{e}{p}\frac{dB_z}{dx}x + \frac{e}{p}\frac{1}{2}\frac{d^2B_z}{dx^2}x^2 + \frac{e}{p}\frac{1}{3!}\frac{d^3B_z}{dx^3}x^3 + \ldots \\ &= \frac{1}{\rho} + kx + \frac{1}{2}mx^2 + \frac{1}{3!}\mathcal{O}(x^3) + \ldots.\end{aligned}$$

Consider a constant energy particle in a fixed dipole field which is executing cyclotron motion. Other particles of the same energy execute cyclotron motion of the same but offset radius. Compared to the reference trajectory, this looks like an oscillation, which we call a **betatron oscillation**. The number of oscillations per turn, known as the **tune**, is 1.

1.2.2 Derivation of Hill's Equations

We will now develop the equations of motion in a linear or circular machine. We will use a curved coordinate system, with this curvature being produced by a local dipole field. The local curvature is denoted ρ, the vertical direction is denoted $\hat{\mathbf{z}}$ and the length along the design trajectory is s. The position vector, $\mathbf{R}$, of a particle in this coordinate system is given by

$$\mathbf{R} = r\mathbf{x} + z\mathbf{z}$$

where

$$r = \rho + x.$$

The curved reference trajectory is normally called the orbit, and the coordinate system moves with a reference particle around the design orbit defined by the dipoles. The coordinates represent deviations with respect to the design (**ideal**) orbit, and we assume those deviations will be small (x is normally measured in millimetres). For coordinates relative to this design orbit we use the position and slope $\frac{dx}{ds} = x'$. Ignoring the longitudinal component of motion, at any point there is a right-handed coordinate system described by (x, z, s)

The velocity, $\mathbf{v}$, is given by the time derivative of the position vector

$$\begin{aligned}\mathbf{v} &= \dot{\mathbf{R}} \\ &= \dot{r}\mathbf{x} + r\dot{\mathbf{x}} + \dot{z}\mathbf{z} + z\dot{\mathbf{z}}.\end{aligned}$$

The vertical axis is unchanging and so $\ddot{\mathbf{z}} = \dot{\mathbf{z}} = 0$, as well as the time-derivative of the x vector being given by $\dot{\mathbf{x}} = \dot{\theta}\mathbf{s}$, hence

$$\mathbf{v} = \dot{r}\mathbf{x} + r\dot{\theta}\mathbf{s} + \dot{z}\mathbf{z}.$$

One then gets the velocity components from the directions as

$$\begin{aligned}v_x &= \dot{r} \\ v_z &= \dot{z} \\ v_s &= r\dot{\theta}.\end{aligned}$$

Furthermore, using the fact that $\dot{\mathbf{s}} = -\dot{\theta}\mathbf{x}$, the acceleration vector is given by the time derivative of the velocity vector:

$$\begin{aligned}\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{R}} \\ &= \ddot{r}\mathbf{x} + 2\dot{r}\dot{\mathbf{x}} + r\ddot{\mathbf{x}} + \ddot{z}\mathbf{z} + 2\dot{z}\dot{\mathbf{z}} + z\ddot{\mathbf{z}} \\ &= \ddot{r}\mathbf{x} + 2\dot{r}\dot{\theta}\mathbf{s} - r\dot{\theta}^2\mathbf{x} + \ddot{z}\mathbf{z} \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{x} + 2\dot{r}\dot{\theta}\mathbf{s} + \ddot{z}\mathbf{z}.\end{aligned}$$

To calculate the force on the particle, we use the Lorentz force law with no electric field and only transverse magnetic fields;

$$\frac{d\mathbf{p}}{dt} = e\mathbf{v} \times \mathbf{B}.$$

Expanding the cross product gives the components of the rate of change of the momentum as

$$\begin{aligned}\frac{d\mathbf{p}}{dt} &= e\begin{vmatrix}\mathbf{x} & \mathbf{z} & \mathbf{s} \\ v_x & v_z & v_s \\ B_x & B_z & 0\end{vmatrix} \\ &= -ev_sB_z\mathbf{x} + ev_sB_x\mathbf{z} + e\left(v_xB_z - v_zB_x\right)\mathbf{s}.\end{aligned}$$

Writing the momentum in terms of the position vector $\mathbf{R}$:

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt} m\gamma\dot{\mathbf{R}} = m\gamma\ddot{\mathbf{R}},$$

where γ has been assumed to be constant as there is only a magnetic field and so cannot do work on the particle. The expression for the rate of change of momentum is given by

$$\frac{d\mathbf{p}}{dt} = m\gamma\left(\left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{x} + 2\dot{r}\dot{\theta}\mathbf{s} + \ddot{z}\mathbf{z}\right).$$

The radial and vertical equations of motion are therefore

$$\begin{aligned}\ddot{r} - r\dot{\theta} &= -\frac{eB_z}{m\gamma}v_s \\ \ddot{z} &= \frac{eB_x}{m\gamma}v_s.\end{aligned}$$

We now want to change the independent variable from time to the spatial distance traversed, s, as this is what we care about and magnets are localised in position and not time. It is incorrect to use

$$\frac{ds}{dt} = v_s,$$

instead there is an extra factor due to the gemoetry of the orbit:

$$\frac{ds}{dt} = v_s\frac{\rho}{\rho + x}.$$

The radial and vertical derivatives can now be converted to the geometry of the orbit:

$$\begin{aligned}\dot{r} &= \frac{dr}{dt} \\ &= \frac{v_s\rho}{\rho + x}\frac{dr}{ds} \\ &= \frac{v_s\rho}{\rho + x}\frac{dx}{ds},\end{aligned}$$

and so

$$\begin{aligned}\ddot{r} &= \frac{d^2r}{dt^2} \\ &= \left(\frac{v_s\rho}{\rho+x}\right)^2 \frac{d^2x}{ds^2}\end{aligned}$$

and

$$\ddot{z} = \left(\frac{v_s\rho}{\rho+x}\right)^2 \frac{d^2z}{ds^2}.$$

We can change the independent variable t to s in the equations of motion we derived from the Lorentz force law by swapping each dt for ds and then use

$$\begin{aligned}r\dot{\theta} &= v_s \\ \Delta S &= \rho\Delta\theta \\ r\dot{\theta}^2 &= \frac{r^2\dot{\theta}^2}{r} \\ &= \frac{v_s^2}{r} \\ &= \frac{v_s}{\rho+x}.\end{aligned}$$

Doing this, we then obtain the equation for x as

$$\left(\frac{\rho}{\rho+x}\right)^2 \frac{d^2x}{ds^2} - \frac{1}{\rho+x} = -\frac{eB_z}{m\gamma v_s}$$

which, by using the Taylor expansions $\left(\frac{\rho}{\rho+x}\right)^2 \approx 1 - \frac{2x}{\rho} + \ldots$ to first order and $\frac{1}{\rho+x} \approx \frac{1}{\rho} - \frac{x}{\rho^2} + \ldots$ to second order, becomes

$$\frac{d^2x}{ds^2} - \left(\frac{1}{\rho} - \frac{x}{\rho^2}\right) = -\frac{eB_z}{m\gamma v_s}.$$

However, from earlier we know that $\frac{1}{\rho} = B_{z0}\frac{e}{p}$ where B_{z0} is the guide field, which when combined with the assumption that $p = m\gamma v \approx m\gamma v_s$;

$$\frac{d^2x}{ds^2} + \frac{x}{\rho^2} = -\frac{e\left(B_z - B_{z0}\right)}{p}.$$

By the same method, the vertical equation of motion is given by

$$\frac{d^2z}{ds^2} = \frac{eB_x}{p}.$$

An important application of the transverse equations of motion is when the magnetic field contains only dipoles and quadrupoles components, thus the magnetic field is given by

$$\mathbf{B} = B_{z0}\mathbf{z} + g(z\mathbf{x} + x\mathbf{z}),$$

where $g = \frac{\partial B_z}{\partial x} = \frac{\partial B_x}{\partial z}$ is the gradient of the magnetic field. Doing this, the guide field term in the horizontal equation of motion, B_{z0}, cancels and so one can split the equation by directionality (vectors) to give

$$\frac{d^2x}{ds^2} + \left(\frac{g}{B\rho} + \frac{1}{\rho^2}\right)x = 0 \text{ and } \frac{d^2z}{ds^2} - \frac{g}{B\rho}z = 0.$$

Note that B_{z0} and g are periodic functions of the independent variable s. This periodicity is either the machine circumference or the length of a repeating period (cell), L. We have therefore derived the equations of motion for transverse dynamics. Note that as the beam is focussed in x the beam defocusses in y due to $\nabla \times \mathbf{B} = 0$.

1.2.3 The Transfer Matrix Approach

After some effort, Hill's equations were derived as linearised second-order differential equations for the transverse variables x and z in dipole and quadrupole fields in a particle accelerator:

$$\frac{d^2x}{ds^2} + \left(\frac{g}{B\rho} + \frac{1}{\rho^2}\right)x = 0 \text{ and } \frac{d^2z}{ds^2} - \frac{g}{B\rho}z = 0.$$

In the horizontal equation, g arrives from the quadrupoles and there is also a natural focussing from the dipoles in the plane of the curvature. Writing $\frac{g}{B\rho}$ as some **piecewise** constant $k(s)$ and ρ as some piecewise constant variable $\rho(s)$, the equation of motion becomes

$$x(s)'' + \left(k(s) + \frac{1}{\rho(s)^2}\right)x(s) = 0.$$

Hill's equations in both planes can be compactly written by denoting the direction x or z with u and writing the position-dependent 'spring constant' as K:

$$\frac{d^2u}{ds^2} + Ku(x) = 0.$$

When $K < 0$ then $u = z$ or if $K > 0$ then $u = x$. The spring constants are then

$$K_x = \frac{1}{B\rho}\frac{\partial B_z}{\partial x} + \frac{1}{\rho^2}$$
$$K_y = -\frac{1}{B\rho}\frac{\partial B_z}{\partial x}.$$

In these equations, ρ comes from the natural horizontal focussing in dipoles and the gradient is the strong-focussing in quadrupoles. In reality, magnets tend to have fields which extend beyond the physical length of the magnet. This is accounted for by defining an **effective length**, l_{eff}, as the length of the magnetic elements as as seen by the beam itself:

$$Bl_{eff} = \int_0^{l_{mag}} \mathbf{B} \cdot d\mathbf{s}.$$

1.2.4 Matrix Properness

1.3 Lattice Functions

1.3.1 Lattice Functions

1.3.2 The Courant-Snyder Formalism

1.3.3 How to Transform Lattice Functions

1.3.4 Stability

1.3.5 Tune

1.4 Optics and Lattice Design

1.4.1 The FODO Cell

1.4.2 Lattice Design

1.4.3 Mini Beta Insertions

1.4.4 Principles of Lattice Design

1.4.5 Optical Structures

1.5 Errors in Lattices

1.5.1 Field Errors

1.5.2 Closed-Orbit Distortion

1.5.3 Tune Shifts

1.6 Real Particles

1.6.1 Dispersion

1.6.2 Momentum Compaction Chromaticity

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Hamiltonian Dynamics

2.1 Review of Newtonian, Lagrangian, and Hamiltonian Mechanics

2.1.1 Canonical Momentum

2.1.2 Hamiltonian Mechanics and Hamilton's Equations

2.1.3 Hamiltonian for a Charge Particle in an EM Field

2.1.4 Symplecticity

2.1.5 Canonical Transformations and Generating Functions

2.2 The Accelerator Hamiltonian

2.2.1 Hamiltonian for a Relativistic Charged Particle in an EM Field

2.2.2 Path Length

2.2.3 Reference Momentum

2.2.4 Energy Deviation and Normalised Momenta

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Colliders

3.1 History

3.2 Recap of Beam Dynamics

3.3 Synchrotron Radiation and Particle Physics

3.4 Beam-Beam Physics and Luminosity

3.5 A Circular Collider: The LHC

3.6 A Linear Collider: The ILC

3.7 Other Colliders: HERA, LEP, LHeC

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Beam Dynamics with Space Charge

4.1 Recap of Transverse Dynamics

4.2 Single Particle Dynamics in an EM Field

4.3 Beam Dynamics with Space-Charge

4.4 Coherent and Incoherent Tune Shift

4.5 Modified Equations of Motion

4.6 Effect of Image Charges

4.7 Effects of Matched and Mismatched Beams

4.8 Non-Linear Beam Dynamics and Space-Charge

4.9 Beam Evolution with Space-Charge

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Lattice Designs

5.1 Transfer Matrices for Periodic Lattices

5.2 Transfer Matrices for Twiss Parameters

5.3 FODO Lattices

5.3.1 Stability Condition

5.3.2 Transfer Matrix

5.3.3 Max-Min Beta Functions

5.4 Dispersion Suppression Lattices

5.5 Different Cell Designs

5.5.1 Chasmann-Green

5.5.2 Double/Triple Bend Achromats

5.5.3 Matching Cells

5.6 Injection/Extraction Cells

5.7 Advanced Matching

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Non-Linear Dynamics

6.1 Nonlinear Dynamics in Bunch Compressors

6.2 Nonlinear Dynamics in Storage Rings

6.3 Hamiltonian Mechanics

6.4 Canonical Perturbation Theory

6.5 Lie Transformations

6.6 Symplectic Integrators

Chapter 7

Circular Machines

7.1 History

7.2 Basic Beam Dynamics

7.3 Synchrotron Radiation

7.4 NINA and SRS

7.5 Technological Challenges at the Energy Frontier

7.6 LEP and LHC

7.7 Next Generation Colliders

7.8 Spallation Sources and ISIS

7.9 Neutrino Factories

7.10 FFAG and EMMA

7.11 Generic Proton Accelerator Developments