Doc Dave

MPhys Notes

Dynamics

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Based on a course by Prof. Chris Parkes and Dr. Neal Jackson

July 4, 2018

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Chapter 1

Linear Dynamics I

1.1 Definitions

A number of the terms defined in this section have other definitions, but for the purpose of this course the reader needs only the definitions given herein. Furthermore, this course will ignore all relativistic effect, i.e. $v \ll c$.

Definition 1.1: Scalars and Vectors

A **scalar** is a quantity which is fully described by a single number, called its *magnitude*, e.g. volume, time, temperature. In contrast, a **vector** is described by a length (magnitude) and a direction (also called its *argument*), e.g. compass navigation, friction.

Vectors are written as sets of lengths for each dimension in which it exists, written between parentheses, e.g. a vector $\mathbf{v}$ on a page which spans 2 units in the x-direction and 4 in the y-direction would be written $\mathbf{v} = (2, 4)$. Alternatively, a vector $\mathbf{u}$ which exists in 3D space and spans 5 units in the x-direction, 3 in the y-direction and 12 in the z-direction would be written $\mathbf{u} = (5, 3, 12)$.

Definition 1.2: Unit Vector

A **unit vector** is a vector of length 1 (also said to be of *unit* length) and direction parallel to a given axis. Unit vectors are denoted as vectors with circumflexes (or *hats*), e.g. the unit vector $\hat{\mathbf{a}}$ is pronounced "a hat". The unit vector $\hat{\mathbf{v}}$ of a non-zero vector $\mathbf{v}$ is given by the vector down-scaled by its magnitude $|\mathbf{v}|$, i.e. $\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$.

By convention $\hat{\mathbf{i}}$ is the unit vector in the x-direction, $\hat{\mathbf{j}}$ is the unit vector in the y-direction, and $\hat{\mathbf{k}}$ is the unit vector in the z-direction.

Definition 1.3: Particle

A **particle** is an object which has had a number of properties removed, e.g. colour. The particle has a location described by coordinates (x, y, z) and it has zero spatial extension, making it a **point-like particle**.

Definition 1.4: Event

An **event** is the result of some physical phenomenon, e.g. collision of two spheres.

Definition 1.5: Distance and Displacement

Distance is a scalar quantity which indicates the length of the path taken by a particle, whereas **displacement**, denoted S, is a vector quantity which indicates both the shortest distance and direction of motion between initial and final positions.

Example 1.1

A particle travels 2 m north, 4 m east, 2 m south, and 4 m west, arriving back to its initial position. The distance covered by the particle is 12 m, however its displacement is $(0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}})$ m.

1.2 Differentiation of Vectors

1.3 Velocity

Definition 1.6: Velocity and Speed

Velocity, denoted $\mathbf{v}$, is the rate of change of position relative to an observer. This quantity is a vector whose magnitude is the **speed**, denoted $|\mathbf{v}|$ or v, of the particle.

The velocity for a particle which moves a distance $\Delta\mathbf{x}$ in a time Δt is given by the ratio

$$\mathbf{v} = \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{\Delta \mathbf{x}}{\Delta t},$$

which one recognises as the derivative of the position with respect to time, i.e.

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\,.$$

1.4 Acceleration

Definition 1.7: Acceleration

Acceleration, denoted $\mathbf{a}$, is the rate of change of velocity relative to an observer, i.e. $\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \equiv \frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}$.

1.5 Inertial Frames

1.6 Newton I

The SUVAT equations are a set of equations describing particles travelling linearly (along a straight path) with constant acceleration. The name SUVAT is an acronym which refers to the variables used: distance travelled s, initial velocity $\mathbf{u}$, final velocity $\mathbf{v}$, acceleration $\mathbf{a}$, and time t. Each equation relates four of the five variables to one another, excluding the fifth. Since the motion is linear, the position, velocity and acceleration are collinear (lie along the same line), and so only the magnitudes of these quantities are required for the equations.

Integrating acceleration $a(t)$ between initial time t_0 and final time t_1 gives

$$\int_{t_0}^{t_1} a(t)\,\mathrm{d}t = \int_{t_0}^{t_1} \frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t = v(t_1) - v(t_0),$$

however since the acceleration is constant the integral is also given by

$$\int_{t_0}^{t_1} a\,\mathrm{d}t = a\left(t_1 - t_0\right).$$

Since the initial and final times are arbitrary, the initial time can be chosen to be at $t = 0$ and the final time at $t = t$. As a result, by relabelling the initial velocity $v(0)$ as u and the final velocity $v(t)$ as simply v, this equation may then be rearranged to give the first SUVAT equation:

$$\boxed{v = u + at.} \tag{1.1}$$

Integrating the left side of this with respect to time gives

$$\int_0^t v(t)\,\mathrm{d}t = \int_0^t \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = x(t) - x(0),$$

which by definition is the displacement of the particle, S. Given that u and a are constants, integrating the right side of the equation then gives

$$\int_0^t u + at\,\mathrm{d}t = ut + \frac{1}{2}at^2,$$

and thus a further SUVAT equation is found:

$$\boxed{s = ut + \frac{1}{2}at^2.} \tag{1.2}$$

Rearranging (1.1) as $a = \frac{v-u}{t}$ and substituting into (1.4), then the third SUVAT equation is

$$\boxed{s = \frac{v+u}{2}t.} \tag{1.3}$$

Now, (1.3) can be rearranged as $t = \frac{2s}{v+u}$ and substituted into (1.1) to give the so-called Torricelli equation

$$\boxed{v^2 = u^2 + 2as.}$$

To derive the final SUVAT equation, rearrange (1.3) as $ut = 2s - vt$ and substitute into (1.4) to give

$$\boxed{s = vt - \frac{1}{2}at^2.} \tag{1.4}$$

1.6.1 Summary

The full set of SUVAT equations are as follows:

$$\begin{aligned} v &= u + at \\ s &= ut + \frac{1}{2}at^2 \\ s &= \frac{v+u}{2}t \\ v^2 &= u^2 + 2as \\ s &= vt - \frac{1}{2}at^2. \end{aligned}$$

The SUVAT equations are only suitable for systems in which the motion is linear and acceleration is constant, however this is often not the case. For example, consider an acceleration which is given in terms of constants α and β as

$$a(t) = -\alpha + \beta t.$$

The velocity is this integrated over time;

$$v(t) = v_0 - \alpha t + \frac{1}{2}\beta t^2$$

and the position is the time-integral of the equation for velocity;

$$x(t) = x_0 + v_0 t - \frac{1}{2}\alpha t^2 + \frac{1}{6}\beta t^3.$$

Chapter 2

Linear Dynamics II

2.1 Newton II

In his 1687 magnum opus *PhilosophiæNaturalis Principia Mathematica*, Isaac Newton laid the foundations for how we now understand objects to behave under certain conditions, which is now known as Newtonian dynamics. As a philosopher of his time, Newton carefully defined the terms which he would refer to prior to using them, to which end he formulated three laws of motion:

First Law	Every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by forces impressed.
Second Law	A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed.
Third Law	To any action there is always an opposite and equal reaction; in other words the actions of two bodies upon each other are always equal and opposite in direction.

These laws appear abstract in a modern scientific setting due to Newton's philosophical background, however they may be easier understood at first in less rigorous but simpler terms:

1. If a force is not applied to a body then its velocity will not change;
2. A net force $\mathbf{F}$ acting on a body of mass m produces an acceleration $\mathbf{a}$: $\mathbf{F} = m\mathbf{a}$;
3. When one body exerts a force on a second body, the second body simultaneously exerts an force equal in magnitude and opposite in direction on the first body.

2.2 Equations of Motion

2.3 Impulse

2.4 Forces

2.5 Action at a Distance

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Chapter 5

Rotational Motion I

5.1 Polar Coordinates

Until this point, the reader will probably have only encountered Cartesian coordinates, in which coordinates of points in a plane are given by distances from two perpendicular axes x and y which meet at an origin $(0, 0)$. However, there are numerous other coordinate systems. Every system can be used for every situation, however the ways in which they describe the system lend themselves useful for different problems. The polar coordinate system measures points in a plane with radial distance r and angle θ. These parameters are taken relative to a central *pole* (where $r = 0$).

In polar coordinates the length of an arc S is related to the radial distance from the origin r and the angle subtended over the arc θ by

$$S = r\theta,$$

where the angle is measured in the **radian** units.

Definition 5.1: Radians

The **radian** is defined as the angle subtended over a distance along the circumference of a circle equal to its radius, i.e.

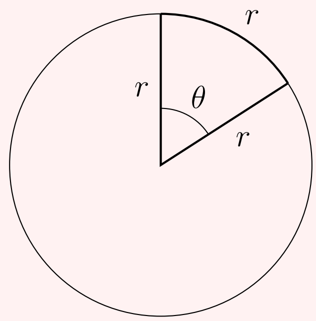

Figure 5.1: Definition of 1 radian.

From its definition, one can see that the total number of radians in a full rotation is 2π.

Proof 5.1: There are 2π radians in a full rotation

The definition of π is the ratio of a circle's circumference C to its diameter d, i.e.

$$\pi \coloneqq \frac{C}{d},$$

where the diameter is defined as twice the radius;

$$d \coloneqq 2r.$$

As such, combining these definition gives

$$C = 2\pi r,$$

and so by comparing this to distance in polar coordinates, $S = r\theta$, one sees that travelling the circumference of a circle subtends 2π radians.

When the Cartesian origin and polar pole coincide, the coordinate systems are related by

$$
\begin{aligned}
x &= r\cos(\theta) \\
y &= r\sin(\theta) \\
r &= \sqrt{x^2 + y^2} \\
\theta &= \arctan\left(\frac{y}{x}\right).
\end{aligned}
$$

5.2 Torque

5.3 Vector Product

The **vector product** is an important concept in angular dynamics.

Definition 5.2: Vector Product

The **vector product** or **cross product**

5.4 Rotation of Coordinate Axes

5.5 Angular Momentum

Chapter 6

Conservation Laws and Isolated Systems

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Chapter 7

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