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MPhys Notes

Mathematics 1

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Based on a course by Dr. Andrew Markwick

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Classically, the gradient of a line is given by the ratio of the change in y, Δy, to the change in x, Δx, i.e.

$$\text{gradient} = \frac{\Delta y}{\Delta x}.$$

As such, a horizontal line has a gradient of zero whereas a vertical line has an infinite gradient. Figure **??** demonstrates how this manifests graphically. This definition of gradient is only a valid description of straight lines, i.e. $y(x) = mx + c$ where m is the gradient and c is some constant. Relationships in physics are rarely this simple and very often extend to higher orders of the independent variable, e.g. $y(x) = ax^2 + bx + c$, and for these situations the classical gradient is not sufficient to describe the gradient at a given point.

In order to generalise the gradient, consider an arbitrary function f which depends upon variable x, $f(x)$, such that it is graphically demonstrated by Figure **??**. The gradient is then given by

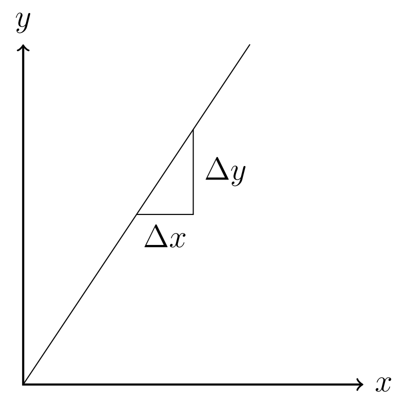

Figure 3.1: Gradient of a straight line.

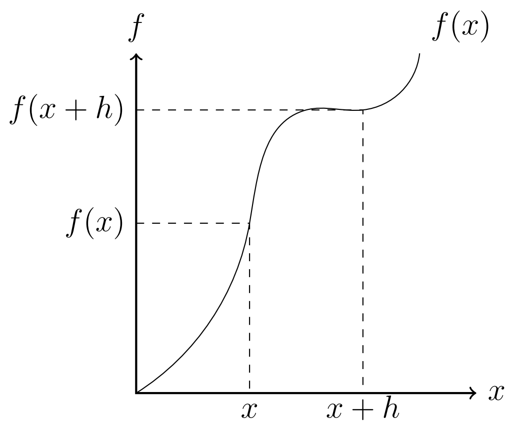

Figure 3.2: Gradient of an arbitrary line.

$$
\begin{aligned}
\text{gradient} &= \frac{\Delta f(x)}{\Delta x} \\
&= \frac{f(x+h) - f(x)}{(x+h) - x} \\
&= \frac{f(x+h) - f(x)}{h}.
\end{aligned}
$$

One can see that as the gap between the points, h, reduces, the classical straight-line approximation becomes applicable, i.e. as h tends to zero the gradient is once again described by the ratio $\frac{\Delta f}{\Delta x}$. The terminology for this shrinking of h is to say that the gradient is "in the limit of h to zero", written $\lim_{h\to 0}$:

$$
\frac{df(x)}{dx} \equiv \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.
$$

This is the defintion of a **derivative**.

3.3 The Product Rule

Using the defintion of derivatives for the product of two functions, $f(x)$ and $g(x)$, gives

$$
\frac{d(f(x)g(x))}{dx} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.
$$

Adding zero to the numerator does not change anything, hence one can write

$$\begin{aligned}
\frac{d(f(x)g(x))}{dx} &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x) + 0}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x) + (f(x+h)g(x) - f(x+h)g(x))}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)(g(x+h) - g(x))}{h} + \lim_{h\to 0} \frac{g(x)(f(x+h) - f(x))}{h} \\
&= \left(\lim_{h\to 0} f(x+h)\right)\left(\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right) \\
&\quad + \left(\lim_{h\to 0} g(x)\right)\left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right) \\
&\quad = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx}.
\end{aligned}$$

The derivative of a function can be written in as the function "primed", i.e. $\frac{df(x)}{dx} = f'$. The **product rule** can therefore be written as

$$(fg)' = fg' + gf'.$$

3.4 The Chain Rule

By definition, the derivative of a function f which varies with another function g, which varies with an independent variable x, $f(g(x))$, is given by

$$\frac{df(g(x))}{dx} = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}.$$

Multiplying by 1 leaves this unchanged, hence one can write this as

$$\begin{aligned}
\frac{df(g(x))}{dx} &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \times 1 \\
&= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \times \left(\frac{g(x+h) - g(x)}{g(x+h) - g(x)}\right) \\
&= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \times \frac{g(x+h) - g(x)}{h}.
\end{aligned}$$

As g is the variable for function f, the limited change in g is the derivative of f, i.e.

$$\frac{df(g(x))}{dx} = f'(g(x)) \times g'(x).$$

This is known as the **chain rule**. An alternative (but far from mathematically rigorous) derivation of this is to consider the infinitesimal changes in terms of a fraction, e.g.

$$\begin{aligned}\frac{\Delta y(x(t))}{\Delta t} &= \frac{\Delta y}{\Delta t} \times \frac{\Delta x}{\Delta x} \\ &= \frac{\Delta y}{\Delta x} \times \frac{\Delta x}{\Delta t} \\ &= y'x'.\end{aligned}$$

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