Doc Dave

PhD Notes

Plasma Physics

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Contents

Chapter 1

Introduction to Plasma Physics

1.1 What is a Plasma?

There are three main definitions of a plasma:

1. (Physics) An assembly of electrically-charged particles;
2. (Haematology) A constituent of blood;
3. (Geology) A type of geological rock.

In 1879 Crookes coined the first definition of plasma as a *fourth state of matter* - an extension to the pre-existing three states of gas, liquid and solid - though not directly naming it a plasma. The term "plasma" was initially used in a physical context by Irving Langmuir in the 1920's due to the amorphous nature of the matter being reminiscent to that of blood plasma. Investigations of electrical discharges date back to the 19$^{\text{th}}$ century, with scientists such as J. J. Thompson being involved in such.

Definition 1.1: Plasma

In modern physics, a plasma is defined as a quasineutral (macroscopically neutral) gas which consists of charged and neutral particles that exhibit collective behaviours.

1.2 Abundance of Plasmas

In day-to-day life we do not intuitively believe that we encounter plasmas very frequently; we breath gaseous air, drink liquid water, eat solid food etc. However, on a universal scale plasmas are actually quite common; they appear as stars, magnetospheres and ionosopheres of planets, nebulae etc. As a result of these, around 99% of all visible matter within the universe is in the plasma state. In fact, even in every day life we encounter plasmas more often that at first glance; flames are often in the plasma state, as are arc discharges produced when pulling an (older) plug out of a socket without first turning it off.

1.2.1 Natural Plasmas

Most natural plasmas occur either in outer space or on the boundary with space, hence include:

- Stars, including our own sun;
- Nebulae;
- Interstellar space contains a very dilute plasma;
- Magnetospheres and ionospheres of planets.

1.2.2 Laboratory Plasmas

Plasmas are extremely useful in a number of laboratory settings, including

- TOKAMAKs used to control thermonuclear fusion reactions;
- Laser-created plasmas;
- Technological plasmas for materials and semiconductor processing.

1.3 Notation and Units

One should note that older (and some contemporary) papers and textbooks use CGS (centimetre-gramme-second) units. It is also useful to note that there are two categories of CGS units: electromagnetic units (EMU) and electrostatic units (ESU) - within each of these there are several subcategories.

Notation used in plasma physics often has super- and subscript letters, such as 'e', 'i', 'n' and 'a', which respectively denote electrons, ions, neutrals, atoms etc. For example, n_e represents the plasma electron number density, which is the number of electrons per unit volume.

Temperatures in plasma physics are referred to in terms of atomic energy units, i.e. electronvolts, eV. The way in which this is calculated is from equating one electronvolt with the thermal energy ($k_B T$);

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J} = k_B \ \left[\mathrm{J\,K^{-1}}\right] \cdot T \ \left[\mathrm{K}\right].$$

As a result;

$$1 \text{ eV} = 11,600 \text{ K}.$$

1.4 Statistics

Electrons and protons both have half-integer spins, and so are fermions, hence in principle a plasma should obey Fermi-Dirac statistics. In practice, however, the occupancy of the states is sufficiently low that most plasmas are well-described by Maxwell-Boltzmann statistics. There are a few exceptions other than cases in which a plasma is in an extreme state, for which Fermi-Dirac statistics must be used. As a reminder, the Maxwell-Boltzmann distribution for velocity is given by

$$f_v\left(v_x, v_y, v_z\right) = \left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} e^{-\frac{mv^2}{2k_B T}},$$

and for speed it is

$$f\left(v\right) = 4\pi v^2 \left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} e^{-\frac{mv^2}{2k_B T}}.$$

From these distributions one can calculate the

- Most probable speed (peak of distribution curve, i.e. derivative = 0):

$$v|_{\frac{df}{dv}=0} = \left(\frac{2k_BT}{m}\right)^{\frac{1}{2}};$$

- RMS speed:

$$\begin{aligned}\sqrt{\langle v^2\rangle} &= \int v^2 f \ dv \\ &= \left(\frac{3k_BT}{m}\right)^{\frac{1}{2}};\end{aligned}$$

- Mean speed:

$$\begin{aligned}\langle v\rangle &= \int v f \ dv \\ &= \left(\frac{8k_BT}{\pi m}\right)^{\frac{1}{2}}.\end{aligned}$$

Proof 1.1: Approximating RMS and mean speeds

For most situations the RMS and mean speeds are very similar values as $\frac{8}{\pi} \approx \frac{9}{\pi} \approx 3$, so being able to quickly derives one gives a decent approximation for the other. By setting the kinetic enery of the particles within the plasma to the thermal energy of the particles, given three degrees of freedom;

$$\frac{1}{2}mv^2 = \frac{3}{2}k_BT.$$

Hence, by approximating the RMS to the square root of the squared velocity:

$$\begin{aligned}\sqrt{\langle v^2\rangle} &\approx \sqrt{v^2} \\ &= \sqrt{\frac{3k_BT}{m}}.\end{aligned}$$

1.5 Ionisation

It is possible to have partially ionised plasmas, with the amount of ionisation being described by the **degree of ionisation**.

Definition 1.2: Degree of Ionisation

The **degree of ionisation** (DoI) is the proportion of ionised particles, n_i, to the total number of particles in a gas, n_t, which is given by

$$\text{DoI} = \frac{n_i}{n_t} = \frac{n_i}{n_a + n_i}.$$

When all atoms are ionised, i.e. $n_a = 0$, the degree of ionisation is equal to 1 or 100%, whereas if no atoms are ionised, i.e. $n_i = 0$, then the degree of ionisation is 0 or 0%.

The Saha ionisation equation, also known as the Saha-Langmuir equation, describes the equilibrium conditions for chemical potentials in that it relates the ionisation state of a plasma to its temperature and pressure. The equation is often written

$$\frac{n_i n_e}{n_a} = \left(\frac{g_i g_e}{g_a}\right)\left(\frac{2\pi m k_B T}{h^2}\right)^{\frac{3}{2}} e^{-\frac{eV_i}{k_B T}}, \tag{1.1}$$

where all symbols take their usual meanings, V_i is the ionisation potential and $g_{i,e,a}$ are the statistical weights of ions, electrons and atoms respectively, which effectively corresponds to the number of available quantum states.

Chapter 2

Time and Length Scales

2.1 Plasma Frequencies

The **plasma frequency** (or electron plasma frequency) is denoted f_p and the angular plasma frequency - often also called the plasma frequency - is denoted ω_p, where

$$\omega_p = 2\pi f_p.$$

Definition 2.1: Plasma Frequency

The plasma frequency is the natural frequency at which electrons within the plasma oscillate.

Theorem 2.1: The plasma frequency ω_p is given by $\omega_p = \sqrt{\frac{n_e e^2}{\epsilon_0 m_e}}$

The differential form of Gauss' law is given by:

$$\nabla \cdot \mathcal{E} = \frac{q\,n}{\epsilon_0} \tag{2.1}$$

and the Lorentz force law for a particle in motion is

$$\mathbf{F} = q\left(\mathcal{E} + \mathbf{v} \times \mathbf{B}\right).$$

Finally, the equation of continuity for a plasma in terms of its density and velocity is given by

$$\partial_t n + \nabla \cdot (n\mathbf{v}) = 0. \tag{2.2}$$

By linearising the number density and velocity into ambient and perturbation components, i.e. of order 0 and 1, respectively;

$$n = n_0 + n_1$$
$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1.$$

Expanding Equation (2.2) and separating according to perturbative order, one obtains three equations:

Order	**Equation**	
0 :	$\partial_t n_0 = 0$	(2.3.1)
1 :	$\partial_t n_1 + \nabla \cdot (n_0\mathbf{v}) = 0$	(2.3.2)
2 :	$\nabla \cdot (n_1\mathbf{v}) = 0.$	(2.3.3)

From these one sees that the equilibrium plasma density is constant (Equation (2.3.1)) and that the perturbation current is uniform (Equation (2.3.3)), however Equation (2.3.2) is a modified continuity equation which can be used to further investigate the dynamics of the system.

Furthermore, by considering the linear and convective accelerations for electrons as a result of a perturbative charged beam, one derives the relationship of forces within a driven plasma as

$$\partial_t \mathbf{v} + (\nabla \cdot \mathbf{v})\,\mathbf{v} = -\frac{e}{m_e}\left(\mathcal{E} + \mathbf{v} \times \mathbf{B}\right). \tag{2.4}$$

By implementing the linearisation and separation methodology to Equation (2.4), one obtains two equations:

Order	**Equation**	
1 :	$\partial_t \mathbf{v} = -\frac{e\mathcal{E}}{m_e}$	(2.5)
2 :	$(\nabla \cdot \mathbf{v})\,\mathbf{v} = -\frac{e}{m_e}\left(\mathbf{v} \times \mathbf{B}\right).$	

Taking the divergence of Equation (2.5) and substituting in Equations (2.1) and (2.3.2) gives

$$\partial_t^2 n_1 = -\frac{n_0\, e^2}{m_e\, \epsilon_0} n.$$

One can now see that the system is a simple harmonic oscillator, of the form

$$\ddot{x} + \omega^2 x = 0.$$

The plasma frequency is therefore given by

$$\omega_p = \sqrt{\frac{n_0 e^2}{m_e \epsilon_0}}.$$

Examples

Plasma	n_e [m^{-3}]	f_p [Hz]
Tokamak	10^{19}	2.8×10^{10}
Glow discharge	10^{15}	2.8×10^{8}
Ionosphere	10^{11}	2.8×10^{6}
Solar wind	5×10^{6}	2×10^{4}

2.2 Cyclotron Frequencies

Plasmas can either have an external applied magnetic field, known as a *magnetoplasma*, or not. Particles in magnetoplasmas oscillate the **cyclotron frequency**.

Definition 2.2: Cyclotron Frequency

The **cyclotron frequency** describes the orbits of a charged particle in a magnetic field.

Theorem 2.2: The cyclotron frequency, ω_c, is given by $\omega_c = \frac{qB}{m}$

For a charged particle to remain moving along a circular trajectory, the centripetal force must be equal to the force exerted by the magnetic field, i.e.

$$\frac{mv^2}{r_L} = |q|\, vB.$$

The **Larmor radius**, r_L, is given by

$$r_L = \frac{mv}{|q|\, B}.$$

Furthermore, calculating the frequency of the orbits by

$$f = \frac{v}{2\pi r}$$

then the cyclotron frequency, ω_c, is

$$\omega_c = 2\pi f = \frac{qB}{m}.$$

This equation can be extended to include relativistic effects, resulting in the cyclotron frequency being given by

$$\omega_c = 2\pi f = \frac{qB}{\gamma m}.$$

Examining the equation for the cyclotron frequency, one sees that increasing the magnetic field increases the frequency at which the particle orbits the field. Furthermore, electrons have much greater cyclotron frequencies than much heavier ions due to their significantly smaller masses.

Examples

There are potentially different ions present in both TOKAMAKs and ionospheres, so for the purpose of simplicity it is assumed that TOKAMAKs use ${}^{2}_{1}D^{+}$ and ionospheres are ${}^{16}_{8}O^{+}$, as well as the particles having energies low enough that the assumption $\gamma = 1$ can be taken.

Plasma	B [T]	f_{ce} [Hz]	f_{ci} [Hz]
TOKAMAK	5	8.8×10^{11}	2.4×10^{8}
Ionosphere	10^{-5}	1.8×10^{6}	60

2.3 Collision Processes in Plasmas

If an electric field, $\mathbf{E}$, is applied to a plasma of electrical conductivity σ then the electrons begin to flow and produce a current density, $\mathbf{J}$, which is given by

$$\mathbf{J} = \sigma \mathbf{E} = \frac{\mathbf{E}}{\eta},$$

where η is the electrical resistivity. It is also possible to relate the current density to the drift velocity of the electrons, $\mathbf{v}_e$, and the drift velocity of the ions, $\mathbf{v}_i$;

$$\begin{aligned} \mathbf{J} &= -n_e e \mathbf{v}_e + n_i e \mathbf{v}_i \\ &\approx -n_e e \mathbf{v}_e. \end{aligned}$$

The current density contributed by the ionic term is usually negligible as the mass of the ions is so much greater than that of electrons and hence $\mathbf{v}_e \gg \mathbf{v}_i$.

Assuming that the electric field is suddeny removed at a time $t = 0$ then, without a field to sustain their velocity, the drifting electrons will gradually slow as a result of collisions within the plasma. The characteristic **time constant**, τ_c, is the electron momentum relaxation time or electron collision time. The collision frequency, ν_c, is given by $\nu_c = \frac{1}{\tau_c}$.

The **mean free path**, λ_{mfp}, is given by

$$\lambda_{\text{mfp}} = v_{\text{rms}} \tau_c.$$

For a plasma to be described as "collisional" then the linear size of the plasma must be greater than the mean free path. If this is not true then the plasma is described as "collisionless".

2.4 Electrical Shielding and the Debye Length

Whilst all plasmas are ionised gases, not all ionised gases are plasmas: an ionised gas has to satisfy all of the **plasma criteria** to qualify for the description of "plasma". For a more qualitative definition of a plasma, the idea of **electrical shielding** must be introduced.

Definition 2.3: Electrical Shielding

Electrical shielding is the reduction in electromagnetic interaction strength due to the blocking of the fields by magnetic or conductive materials.

Dutch physicist Peter Debye studied how charged ionic species, such as electrolytes, behave in fluids. He derived an expression for the distance over which an electric field may be experienced in a plasma, now known as the **Debye shielding distance**, denoted λ_D. What this distance amounts to it that is a charge is immersed into a plasma, that charge will only interact with other charged particles within a certain volume, known as the **Debye sphere**, which is of radius λ_D.

The condition for an ionised gas to be described as a plasma is that the Debye radius is much less than the linear dimension of the plasma, L, i.e:

$$\lambda_D \ll L.$$

Theorem 2.3: The Debye length, λ_D, for a plasma of density n_0 is given by $\lambda_D = \sqrt{\frac{\epsilon_0 k_B T}{e^2 n_0}}$

Consider a positive point charge placed inside a plasma of density n_0. For simplicity, consider a hydrogen plasma such that $n_e = n_i$. As the electron mass is much less than that of the ions, the electrons accelerate much faster than the ions and so it is assumed that the electrons move until reaching an equilibrium and the ions remain stationary. The point charge creates an electrostatic potential, $\Phi(r)$, which is spherically symmetric and so it has no angular dependence.
At this point there is an unperturbed plasma of density n_0 with perturbed electrons of density $n_0 + n_1$, where n_1 is known as the pertur-

bation density. The Boltzmann factor, $e^{-\frac{E}{k_BT}}$, for the plasma electrons is altered by including the potential, such that now

$$\begin{aligned} n_e = f(\mathbf{r}, \mathbf{v}) &= e^{-\left(\frac{\frac{1}{2}mv^2 - e\Phi}{k_BT}\right)} \\ &= e^{-\frac{mv^2}{2k_BT}} e^{\frac{e\Phi}{k_BT}} \\ &= n_0 e^{\frac{e\Phi}{k_BT}}. \end{aligned}$$

As expected, $\lim_{\Phi\to 0} n_e \to n_0$. Using $\nabla \cdot \mathbf{E} = -\nabla^2\Phi$, Poisson's equation can be written

$$\begin{aligned} -\nabla^2\Phi &= \frac{\rho}{\epsilon_0} \\ -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) &= \underbrace{\frac{e}{\epsilon_0}n_0\left(1 - e^{\frac{e\Phi}{k_BT}}\right)}_{\text{Plasma}} + \underbrace{q\delta(\mathbf{r})}_{\text{Point charge}} \quad . \end{aligned}$$

The solution to this is non-linear, so simplification is ideal. As the potential is expected to fall off rapidly with distance from the point charge - much quicker than than $\frac{1}{r}$ dependence in a vacuum - due to shielding from the plasma electrons, the exponential can be expanded to get

$$\begin{aligned} -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) &= \frac{en_0}{\epsilon_0}\left(1 - \left[1 + \frac{e\Phi}{k_BT} + \frac{1}{2}\left(\frac{e\Phi}{k_BT}\right)^2 + \dots\right]\right) \\ &\approx \frac{en_0}{\epsilon_0}\frac{e\Phi}{k_BT} \\ &= \frac{e^2n_0}{\epsilon_0 k_BT}\Phi. \end{aligned}$$

The coefficient of the potential has physical dimensions m^{-2}, hence the inverse-square of the quantity is defined as the Debye length, λ_D;

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_BT}{e^2n_0}}.$$

Hence, in order for collective effects to dominate the behaviour of the plasma then there needs to be many particles within a Debye sphere (high density). One should note that the potential used for the Debye length is subject to two conditions:

$$\lim_{r \to 0} \Phi \to \frac{-en_1}{4\pi\epsilon_0 r},$$
$$\lim_{r \to \infty} \Phi \to 0.$$

The Poisson equation for a charge in a plasma is given by

$$-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) = \frac{\Phi}{\lambda_D^2},$$

the solution for which is

$$\Phi = \frac{-en_1}{4\pi\epsilon_0 r} e^{-\frac{r}{\lambda_D}}.$$

For a classical **weakly coupled** plasma $n_{e,i}\lambda_D^3 \gg 1$, however certain types of plasma can exist without this definition. For example, dust particles in plasma can violate this definition, hence the concept of **strongly coupled** plasmas is introduced.

Definition 2.4: Coupling Parameter

The coupling parameter, Γ, is defined as the ratio of the Coulomb energy to the thermal energy, i.e.

$$\Gamma = \frac{E_{\text{coul}}}{E_{\text{therm}}} = \frac{\left(\frac{q^2}{4\pi\epsilon_0 \langle r \rangle}\right)}{k_B T}.$$

A weakly coupled plasma has $\Gamma \ll 1$, whereas strong coupled plasmas have $\Gamma \gg 1$.

2.5 Boundary Sheaths

Definition 2.5: Boundary Sheath

When a plasma is bounded by a solid surface then there is a **boundary sheath** formed in the region between the plasma and the solid, as shown in Figure 2.1. The sheath arises because the electrons have higher average speeds than the ions, hence the flux of the electrons onto the boundaries tends to be initially greater than that of the ions. The surface becomes a cathode as it acquires a negative electrical potential compared to the plasma, thus electrons are reflected back into the plasma such that the the flux of positive and negative charges are equalised. Positively charged ions are accelerated through the sheath's potential difference resulting in significant energy (many times thermal energy) being directed onto the solid surface. This is the basis of plasma etching of materials, which is preferred over chemical etching for a number of applications due to it being more anisotropic and able to be used on smaller, more fragile objects, e.g. integrated circuits. The thickness of the sheath region is typically a few Debye lengths.

2.5.1 Bohm Sheath Theory

Bohm sheath theory assumes a planar geometry and a negative floating electrode due to asymmetric charging. Ions are assumed to be stationary (cold) compared to the electrons and enter the sheath from a neutral presheath travelling at a speed $\mathbf{v}_s$ toward the wall due to a weak presheath electric field. However, the ions are assumed to have no collisions within the sheath itself and the electrons are assumed to have a Maxwellian distribution, T_e. The edge of the sheath is conventionally taken to be at $x = 0$ and the wall is at $x = d$, hence the width of the sheath. The ion density falls as velocity increases toward the negatively charged electrode and also the electron density falls due to repulsion by the potential of the electrode. The boundary between the sheath and presheath is defined to be at a potential difference of 0 V.

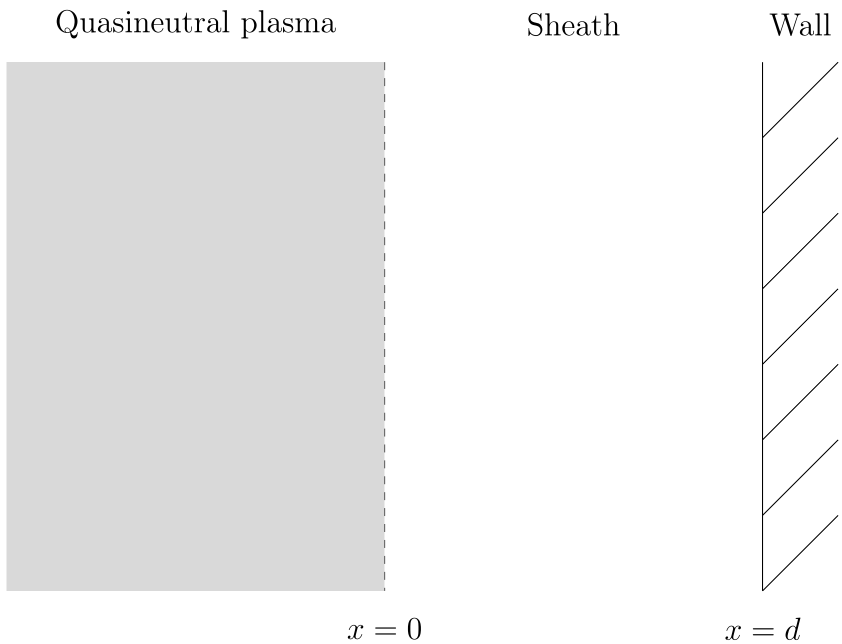

Figure 2.1: Layout of plasma sheath against a solid surface (wall).

The continuity equation for ions is given by

$$n_i\left(x_a\right)\mathbf{v}_i\left(x_a\right) = n_i\left(x_b\right)\mathbf{v}_i\left(x_b\right), \tag{2.6}$$

and the equation for energy conservation is given by

$$\frac{1}{2}m_i v_i^2\left(x_a\right) + e\Phi\left(x_a\right) = \frac{1}{2}m_i v_i^2\left(x_b\right) + e\Phi\left(x_b\right).$$

By definition, along the boundary sheath, i.e. $x_a = 0$, the potential is $\Phi(x_a) = 0$. The velocity of particles entering the sheath is given by $v_i(x_a) = v_s$ and so by dividing by the ion kinetic energy, $E_s = \frac{1}{2}m_i v_s^2$;

$$v_i(x_b) = v_s\left(1 - \frac{e\Phi}{E_s}\right)^{\frac{1}{2}}.$$

Using the Maxwell-Boltzmann distribution for the potential on electrons, the Poisson equation becomes

$$\frac{d^2\Phi}{dx^2} = \frac{\left(n_e(0)e^{\frac{e\Phi}{kT_s}} - n_i\right)}{\epsilon_0}e.$$

Substituting the expression for n_i from Equation (2.6) gives

$$\frac{d^2\Phi}{dx^2} = \frac{n(0)e}{\epsilon_0}\left(e^{\frac{e\Phi}{kT_e}} - \left(1 - \frac{e\Phi}{E_s}\right)^{-\frac{1}{2}}\right)$$

which, according to the chain rule, is

$$\int \frac{d\Phi}{dx}\frac{d^2\Phi}{dx^2}\, dx = \frac{1}{2}\left(\frac{d\Phi}{dx}\right)^2.$$

The solution is then

$$\begin{aligned}\frac{1}{2}\left(\frac{d\Phi}{dx}\right)^2 &= \int \frac{n(0)e}{\epsilon_0}\left(e^{\frac{e\Phi}{k_BT_e}} - \left(1 - \frac{e\Phi}{E_s}\right)^{-\frac{1}{2}}\right)\, dx \\ &= \frac{n(0)e}{\epsilon_0}\left(\frac{k_BT_e}{e}\left(e^{\frac{e\Phi}{k_BT_e}} - 1\right) + \frac{2E_s}{e}\left(\left(1 - \frac{e\Phi}{E_s}\right)^{\frac{1}{2}} - 1\right)\right).\end{aligned}$$

This non-linear differential equation for the potential is known as the **sheath equation**. This cannot be solved analytically but when the potential is small one may use the MacLaurin expansions for e^α and $(1-\alpha)^{\frac{1}{2}}$ to simplify the problem. The equation may therefore be written

$$\left(\frac{d\Phi}{dx}\right)^2 = \frac{\Phi^2}{\lambda_D^2}\left(1 - \frac{k_BT_e}{2E_s}\right),$$

where λ_D is the bulk Debye screening length. The square of the potential gradient must be positive, so the kinetic energy of the ions at the sheath edge must exceed the electron thermal energy.

There are four main regions in question for this theory:

- Solid material;
- Sheath;
- Presheath;
- Plasma.

Due to small charge imbalances, there is a small electric field in the presheath region, which results in a potential difference between the plasma and the sheath-presheath boundary. The minimum value of this potential, V_0, is given by the Bohm sheath criterion as

$$|eV_0| \geq \frac{1}{2}k_B T_e.$$

The ions reach the sheath-presheath boundary from the plasma with this energy and then continue to be accelerated through the sheath region to impinge on the solid material of the wall with typical energies of many times $k_B T_e$. The practical consequence of this is that in order to produce ions with greater energy, one must increase the plasma electron energies.

2.5.2 Models of Plasmas

There are a number of models for plasmas, including:

- **Single particle model**
  This method was used for earlier work in order to discuss very simple motions of electrons and ions in a magnetic field
- **Fluid Model**
  For this model the plasma is treated as a fluid. This therefore implements similar techniques to hydrodynamics however it includes electrical conductivity and a magnetic field, becoming the field of magneto-hydrodynamics (MHD).
- **Kinetic Theory**
  The kinetic theory of plasmas is similar to that of gases but the forces includes the Lorentz force. Typically the plasma kinetic theory describes the plasma via the evolution of the distribution functions of the species of particles present in the plasma.

2.5.3 Examples

Plasma	n_e [m^{-3}]	T_e [K]	λ_{De} [m]
TOKAMAK	10^{19}	10^8	2.2×10^{-4}
Ionosphere	10^{11}	10^3	6.9×10^{-3}

Chapter 3

Discharge and Electromagnetic Processes

3.1 Discharge Processes

3.1.1 Processes That Increase Ionisation

1. Electron collisions with neutral atoms in the ground state to provide ionisation:
$$X + e^- \rightarrow X^+ + 2e^-.$$

2. Two-step ionisation via an intermediate excited state:
$$X + e^- \rightarrow X_m^* + e^-$$
$$X_m^* + e^- \rightarrow X^+ + 2e^-.$$

3.1.2 Processes That Decrease Ionisation

1. Recombination:
$$X^+ + e^- \rightarrow X + \gamma.$$

2. Diffusion of electrons and ions away from regions of ionisation.

3.1.3 Processes That Do Not Change Ionisation

1. Elastic scattering of electrons by neutrals in ground state:

$$X + e^- \rightarrow X + e^-.$$

2. Collisional excitation:

$$X + e^- \rightarrow X^* + e^-.$$

3. Photo-absorption:

$$X^* + \gamma \rightarrow X^{**}.$$

4. Stimulated emission:

$$X^{**} + \gamma \rightarrow X^* + 2\gamma.$$

3.2 Surface Effects

It is usually the same types of particles that are important in collisions with solid surface as within the plasma itself, e.g. ions, electrons, metastable (excited) atoms and photons (particularly UV photons):

1. The photoelectric effect:

$$\gamma + \text{surface} \rightarrow e^- + \text{surface}.$$

2. Positive ions impinging on a cathode, K;

$$X^+ + K \rightarrow X + e^- + K.$$

   - As the atom has already been ionised the electron is secondary and attains an energy $e(V_i - \phi)$, where V_i is the ionisation potential, i.e. the potential energy taken to the surface by the ion, and ϕ is the work function of the surface material.

- Aside: Very energetic positive ions can lead to cathode "sputtering", whereby groups of atoms can be ejected. This can be employed in applications such as reactive ion etching and coating of mirrors. Energetic electrons can also scatter secondary electrons from the anode, though not typically from the cathode.

3. Metastable atoms with the cathode:

$$X^* + K \to X^+ + e^- + K.$$

   - In this case the electron attains an energy $e(V^* - \phi)$, where V^* is the metastable excitation energy.

4. The Schottky effect: The potential barrier at the surface is reduced by an applied electric field and electrons escape over the top of the barrier (thermionic emission) - this effect is strongly enhanced by heating.

5. Field emission: A very large electric field ($\sim 10^{10}$ V m^{-1}) at a point narrows the potential barrier and the electrons escape by quantum tunnelling.

3.3 Electrical Breakdown of Gases

A common way of forming a plasma is to provide electrical energy to a gas, which can be done at a range of frequencies, from DC to ionising radiation. For non-ionising EM frequencies this process is usually a staged process exploiting collisions. Figure 3.1 shows the regions of discharge, which follows a complicated resistive behaviour which will be later quantified. Such discharges are used for certain types of lasers and vapour and fluroescent lamps, as well as being present in lightning.

3.4 Elastic Collisions

When two particles come within close proximity of one another along some axis, they transfer energy as a function of the collision angle. The fraction

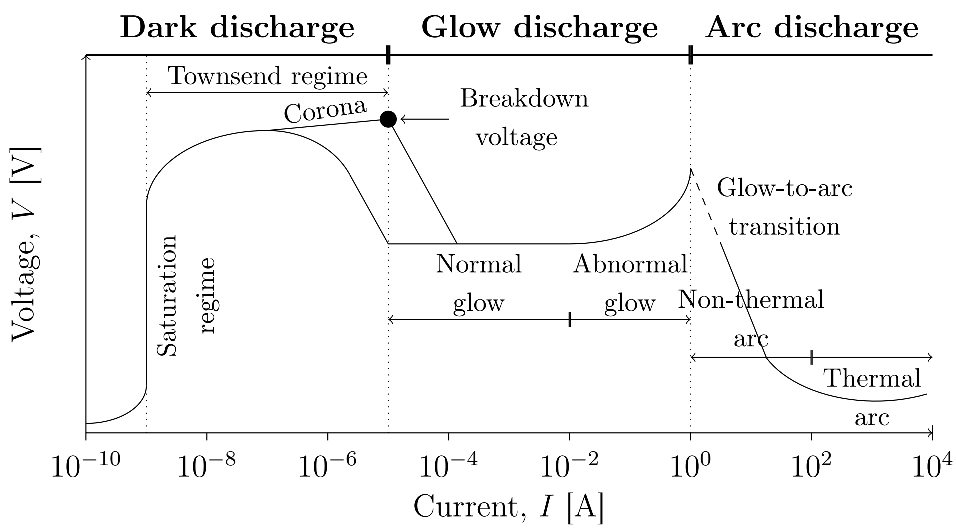

Figure 3.1: Characteristic curve for discharge of a gas.

of energy exchanged, δ, is given by

$$\delta = \frac{\Delta E}{E} = \frac{4m_1 m_2}{(m_1 + m_2)^2} \cos^2(\theta).$$

Therefore a head-on collision gives maximal energy transfer whereas a glancing collision gives a very small transfer. The probability of a collision at any given angle is given by the area presented normal to the axis along which they collide:

$$\begin{aligned} P &= (2\pi r \sin(\theta))\,(r d\theta \cos(\theta)) \\ &= \left(\pi r^2 \sin(2\theta)\right)\ d\theta. \end{aligned}$$

The average energy transfer, $\overline{\delta}$, can be derived by integrating the transfer fraction over all possible angles of incidence multiplied by the probability of that collision angle and divided by the total cross section, i.e.

$$\begin{aligned} \overline{\delta} &= \frac{\int_{\theta=0}^{\theta=\frac{\pi}{2}} \left(\frac{4m_1 m_2}{(m_1+m_2)^2} \cos^2(\theta)\right) (\pi r^2 \sin(2\theta))\ d\theta}{\pi r^2} \\ &= \frac{2m_1 m_2}{(m_1 + m_2)^2}. \end{aligned} \tag{3.1}$$

Notice that if $m_1 = m_2$ then $\overline{\delta} = \frac{1}{2}$, whereas if $m_1 \ll m_2$ then $\overline{\delta} = \frac{2m_1}{m_2}$, so for hydrogen $\overline{\delta} \sim 0.1\%$.

3.5 Mobility

The change in momentum following a collision is given by the impulse of the force:

$$\begin{aligned} m_{i,e}\overline{\mathbf{v}}_{i,e} &= \frac{e\mathbf{E}}{\nu_c} \\ \Rightarrow \overline{\mathbf{v}}_{i,e} &= \frac{e\mathbf{E}}{m_{i,e}\nu_c}, \end{aligned}$$

where $\nu_c = \frac{v_{th}}{\lambda_{\text{mfp}}}$ is the elastic collisional frequency, where v_{th} and λ_{mfp} are the mean thermal speed and mean free path, respectively. These parameters differ for ions and electrons since the effective cross section and closing speeds differ by a lot. The proprtionality constant linking the drift velocity and the electric field is called the **mobility**, denoted $\mu_{i,e}$;

$$\mu_{i,e} = \frac{e}{m_{i,e}\nu_c}.$$

3.6 Scattering Cross Sections

The mean free path, λ_{mfp}, is related to the mean free time, τ_c, by

$$\lambda_{\text{mfp}} = v_{th}\tau_c$$

and the mean free time is related to the gas density, n_g, and collisional cross section, σ, by

$$\begin{aligned} \tau_c &= \frac{1}{n_g\sigma v_{th}}, \\ \Rightarrow \lambda_{\text{mfp}} &= \frac{1}{n_g\sigma}. \end{aligned}$$

It should be noted that σ is usually a function of energy.

If the collisions between electrons and ions/atoms are considered then σ is defined purely by the larger particle. When the particles are of similar sizes then the effective cross section for collisions is increased by a factor of 4. For particles with similar speeds, the mean closing velocity is also higher at $\sqrt{2}v_{th}$ (assuming thermal speeds dominate).

3.7 Inelastic Collisions

Inelastic collisions involve a conversion of particle kinetic energy to potential energy, which can be in the form of ionisation or excitation of an atom, dissociation of a molecule etc. These collisions are characterised by a minimum energy, exceeding which electrons can contribute to the cross section and below which the cross section is taken to be zero. Furthermore, both electron density and cross section vary with energy in a complex way in that the distribution function is primarily controlled by the temperature, whereas the cross section depends on the species of gas. In spite of this complex relationship, a collisional rate coefficient can be defined:

$$
\begin{aligned}
\frac{dn}{dt} &= \int_{v_T}^{\infty} \sigma(v_e) v_e n_a \; dn_e(v_e) \\
&= \int_{v_T}^{\infty} \sigma(v_e) v_e n_a n_e f(v_e) \; dv_e \\
&= \int_{v_T}^{\infty} \sigma(E_e) \left(\frac{2E_e}{m_e}\right)^{\frac{1}{2}} n_a n_e f(E_e) \; dE_e \\
&= n_a n_e \int_{v_T}^{\infty} \sigma(E_e) \left(\frac{2E_e}{m_e}\right)^{\frac{1}{2}} f(E_e) \; dE_e \\
&= n_a n_e K,
\end{aligned}
$$

where K is some constant. At low temperatures the rate is a very strong function of temperature, hence the only electrons that count are those in the tail of the distribution. Often the tail electrons are depressed below the Maxwellian, even when the bulk itself is Maxwellian.

3.8 Elastic Energy Balance

The average amount of energy loss by a particle in a collision may be equated to the work done by an electric (accelerating) field in one mean free path:

$$\delta \cdot \frac{1}{2} m v_{th}^2 = \delta \cdot \frac{3}{2} k_B T = E e \lambda_{\mathrm{mfp}} = \frac{Ee}{nq}.$$

If the electric field and pressure are varied such that their ratio is kept constant, the electron temperature - and thus the inelastic rate - will not be severely affected. This ratio is known as the **reduced electric field**.

Since the electrons have a higher mobility, higher drift velocity, and longer mean free path, the rate of work for them is higher and so the tendency is for electrical work to heat electrons within the plasma and hence collisions mostly convert electric work to electron thermal work.

The average energy delivered to an electron per unit time is given by

$$\begin{aligned} E e u_e &= E e \mu_e E \\ &= E^2 e \mu_e \\ &= E^2 e \left(\frac{e}{m_e \nu_c} \right) \\ &= \frac{E^2 e^2}{m_e \nu_c}. \end{aligned} \tag{3.2}$$

The energy lost by the electron per unit time by elastic collisions is given by

$$\frac{3}{2} k_B (T_e - T_g) \nu_c \delta, \tag{3.3}$$

where δ is the average fraction of energy exchanged by an elastic collision given by Equation (3.1) and $T_{e,g}$ are the electron and gas temperatures, respectively. Equations (3.2) and (3.3) give that

$$T_e - T_g = \frac{3}{2} \frac{E^2 e^2}{k_B m_e \nu_c^2 \delta},$$

hence at low pressures the electron-atom collision frequency ν_c is small and in the denominator it is also squared, so $T_e - T_g$ will be large.

3.9 Avalanche Breakdown: The Townsend Criterion

When an anode and cathode are placed inside an ionised gas, they form an accelerating field. Free electrons are accelerated toward the anode, which allows it to ionise an atom and liberate another electron. Both electrons are then accelerated toward the anode, each of which can ionise their own atoms and a potential 4 free electrons may then exist. This process of increasing the number of free electrons is called an **avalanche** and the minimum number of electrons freed over a number of interactions in order to sustain the avalanche is described by the **Townsend criterion**.

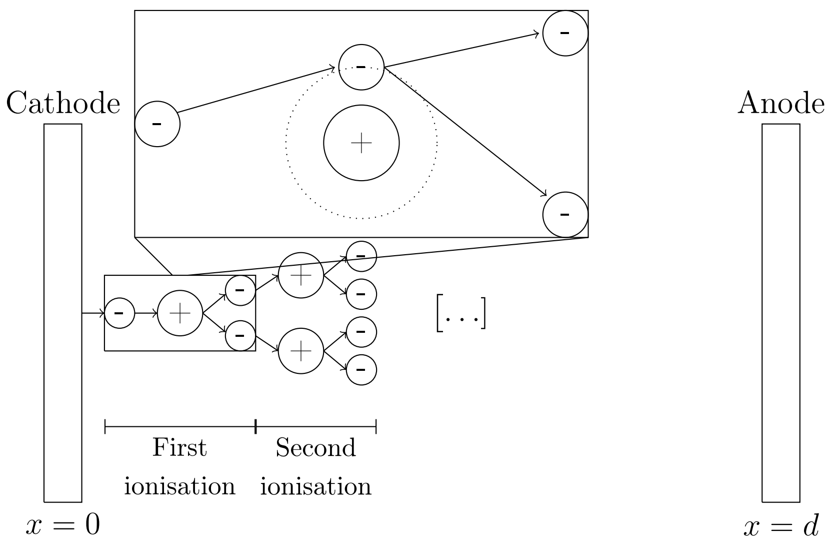

Figure 3.2: Cathode-anode system demonstrating the avalanche effect.

Consider the average number of new electron-ion pairs formed by collisional ionisation of one electron in a unit path length from the cathode to the anode;

$$dn_e = \alpha n_e(x)\ dx, \tag{3.4}$$

where α is the **first Townsend ionisation coefficient** - otherwise known as the **first Townsend avalanche coefficient** - which is the normalised probability of ionising collisions. Hence along the path from cathode to anode the number of electrons increases:

$$n_e(d) = n_e(0)e^{\alpha d} \text{ and}$$
$$I_e(d) = I_e(0)e^{\alpha d},$$

where d is the cathode-anode gap spacing and $n_e(0)$ is the density of electrons emitted from the cathode, perhaps by photo- or thermionic emissions. The total number of electrons which must arrive at the anode is $n_e(d) - n_e(0)$, i.e. the number of electrons that have been "bred" in the discharge. The number of ions bombarding the cathode, n_+, must be the same as the excess electrons at the anode, i.e.

$$n_+(0) = n_e(d) - n_e(0) = n_e(0)\left(e^{\alpha d} - 1\right),$$

where $e^{\alpha d} - 1$ is the **ion multiplication factor**. Suppose that each ion ejects γ electrons by secondary emission from the cathode, then

$$n_{2e}(0) = n_e(0)\gamma\left(e^{\alpha d} - 1\right),$$

where γ is known as the **second Townsend ionisation coefficient**. If this $\gamma\left(e^{\alpha d} - 1\right) > 1$ then every primary electron emitted from the surface will generate at least enough descendant ions to produce one further electron due to bombardment. Such a discharge is self-igniting and self-sustaining - sooner or later a random electron will be formed close enough to the cathode, and from there the discharge will continue forever, hence the **Townsend breakdown criterion** is that

$$\gamma\left(e^{\alpha d} - 1\right) \geq 1.$$

3.9.1 Current in the Sub-Critical Breakdown

Current is proportional to the carrier density, hence the cathode electron current is the sum of primary and secondary currents:

$$I_e(0) = I_{1e}(0) + I_{2e}(0).$$

Note that there is also an ion current at the cathode, so this is NOT the total cathode current. The secondary electron current can be written in terms of the total electron current at the cathode:

$$I_{2e}(0) = I_{1e}(0) + I_{2e}(0)\gamma\left(e^{\alpha d} - 1\right)$$
$$\Rightarrow I_{2e}(0) = \frac{I_{1e}(0)}{1 - \gamma\left(e^{\alpha d} - 1\right)}.$$

The total current can then be found by assessing the electron current at the anode as there are very few ions at the anode:

$$\begin{aligned} I_e(d) &= I_e(0)e^{\alpha d} \\ &= \frac{I_{1e}(0)e^{\alpha d}}{1 - \gamma\left(e^{\alpha d} - 1\right)} \\ &\approx I(d) = I(0). \end{aligned}$$

3.9.2 First Townsend Ionisation Coefficient

John Townsend, after whom the avalanche coefficients were named, proposed a semi-empirical formula for the first ionisation coefficient of the form

$$\frac{\alpha}{P} = Ae^{-\frac{PB}{E}},$$

where A and B are arbitrary coefficients, P is the plasma pressure and E is the plasma energy. In concept the ionisation rate coefficient is given by

$$\frac{dn_e}{dt} = n_a n_e \int_0^\infty q_i(E) v_{th} f(E)\ dE.$$

Since $dt = \frac{dz}{v_e}$ then $\frac{dn_e}{dt}$ transforms to $v_e \frac{dn_e}{dz}$, hence

$$\frac{1}{n_e}\frac{dn_e}{dz} = \frac{n_a}{u_e}\int_0^\infty q_i(E) v_{th} f(E)\ dE.$$

According to Equation (3.4) the LHS of this is α, which one then normally normalises to the gas number density, n_a;

$$\frac{\alpha}{n_a} = \frac{1}{v_e}\int_0^\infty q_i(E)\left(\frac{2E}{m}\right)^{\frac{1}{2}} f(E)\ dE.$$

However, the mean free path of an electron is given by $\lambda_e = \frac{1}{n_a \sigma}$, where σ is the total collisional cross section, then

$$\alpha = \frac{1}{v_e \lambda_e} \int_0^\infty \left(\frac{\sigma_i(E)}{\sigma} \right) \left(\frac{2E}{m} \right)^{\frac{1}{2}} f(E)\, dE,$$

so typically α is expected to take a value proportional to the pressure multiplied by a function of temperature, gas ionisation and elastic cross section.

3.9.3 Determining α and γ

Considering a variable plane parallel gap, the current can be measured as a function of gap spacing as the gap is allowed to become small and finding the limit $\lim_{d \to 0} i_0$. Note that for this measurement the ratio $\frac{E}{P}$ must be kept constant, i.e. the voltage must drop. Neglecting secondary electrons from the cathode, which is typically true for low electric fields and low current levels, the experimental measurement will tend to

$$\ln \left(\frac{i}{i_0} \right) = \alpha d.$$

Plotting $\ln \left(\frac{i}{i_0} \right)$ against d will give α as the gradient, however as I or V increases there is a deviation from the curve, which can be used to obtain γ. By repeating the experiment at different P and $\frac{E}{P}$ both unknown Townsend coefficients, A and B, can be determined. One should be wary that γ is not a simple constant as it depends on T_e - and hence $\frac{E}{P}$ - as well as the sheath voltage drop, cathode material and gas composition.

3.10 Paschen's Law

Substituting Townsend's form for α into the breakdown condition:

$$V_B = \frac{B_{Pd}}{\ln(Pd) + \ln \left(\frac{A}{\ln(1+\frac{1}{\gamma})} \right)}.$$

The result of this is that if γ is constant then the breakdown voltage is a function of only the product of the pressure and gap spacing, which is known as **Paschen's Law**. This equation is valid close to the minimum of the **Paschen Curve**.

3.11 AC Operation and Breakdown

Plasmas have been demonstrated as being useful for materials processing; in industrial processing one may wish to bombard insulators as well as conductors to achieve sputtering or etching. This is done by placing the insulator on one of the electrodes - normally one would choose to make this the cathode since the effect of the ions on the material to be processed is greater than that of the electrons.

When the switch is closed the negative potential on the cathode results in a field across the insulator and gap, so the discharge ignites. The ions bombard the surface of the insulator and the insulator charges up as more and more voltage appears across the dielectric. Less voltage appears across the discharge and if the reduced electric field falls far enough then the discharge will extinguish. The solution to this is to operate in an alternating current (AC) mode with a frequency such that it oscillates quicker than the time for the "capacitor" to charge, then the electron current density in the reverse cycle neutralises the surface charge on the insulator.

3.11.1 Particle Transport

The distance moved by a particle in a sinusoidally-varying field is given by

$$x = \int_0^t \mu E_0 \cos(\omega t)\ dt = \frac{\mu E_0}{2\pi f},$$

where t has been assumed to be one-quarter of a cycle. For the ions, if this distance is greater than the gap spacing then one can expect no significant perturbation in the discharge behaviour when compared to the direct current (DC) conditions. Note that the frequency and/or gap spacing at which DC conditions may be assumed increases with the amplitude of the

AC field. When the fields are reversed the ions have a half-cycle to cross the discharge gap, which yields a critical frequency at which ions become "long-term" trapped in the discharge cavity, resulting in progressive increases in the positive space charge in the gap. The critical frequency, f_c, is given by

$$f_c = \frac{\mu_+ E_0}{\pi d}.$$

However, electrons do not become trapped in the discharge until a frequency significantly above this as their mobility is much greater;

$$f_{ce} = \frac{\mu_e E_0}{\pi d}.$$

At frequencies above this the electrons are no longer lost by their mobility to the electrodes but instead by their diffusion to the walls.

3.11.2 Frequency Response

At RF frequencies electron loss is usually by mobility, though ion loss can be due to diffusion. At microwave frequencies it is typical for the loss mechanism of both species to be diffusion. As the frequency increases, ions stop reaching the cathode, hence the second Townsend process weakens and V_B increases. However, as trapped ions increase the positive space charge of plasma there is increase in electron emissions. As the frequency further increases, electrons are trapped in a column, which enhances the first Townsend process and lengthens the paths and so results in a decrease in V_B, hence the electrodes become no longer necessary. At even higher frequencies the AC field may vary at a rate which is higher than the electron collisional frequency, greatly reducing the numbers of collisions within the plasma and reducing heating, hence having higher breakdown voltages.

3.12 RF Self-Bias

The application of an RF power supply to produce electrons that impinge upon an electrode such that it acquires a negative electrical charge is known as **self-bias** because it cannot discharge to ground electrical levels.

3.12.1 Pros of RF

There are lower losses of charged particles as the power is delivered to the discharge by displacement currents rather than physical currents, leading to higher efficiency. Furthermore, self-bias means that the bombardment of the target by the ions is nearly continuous in spite of the AC variation of the field. Higher temperatures can be obtained in RF (as opposed to AC or DC drives), which is helpful in production of free radicals and other aggressive chemical agents - this helps in production of ionisation and dissociation events. Finally, RF discharges are usually more reliable and suffer fewer instabilities.

3.12.2 Cons of RF

RF components are usually more complex, requiring many more parts, which potentially introduces reliability issues. Finally, it is more difficult to accurately model the discharge process due to poor data and difficult modelling.

3.12.3 Limitations of RF Discharge

It is not always possible to vary the RF frequency, possibly because the oscillator of the power supply is fixed or because the frequency ranges needed are legally controlled, e.g. 13.56 MHz, 2.45 GHz. It may be possible to adjust the pressure, which varies with the collisional frequency, moving it with respect to the drive frequency to obtain the desired power transfer. The maximum power transfer occurs at the resonance frequency, $\omega = \nu_c$, although other resonances may occur if the discharge is magnetised due to cyclotron effects, although it is not necessarily the case that optimum power coupling is the optimum operating condition.

3.13 Problems

1. In a discharge where α is 1.1 cm^{-1} and γ is 0.02, estimate the gap spacing required for a self-sustained discharge.

2. If the gap spacing is 3 cm and the primary electron emission current (thermionic) is 20 pA, estimate the total current in the discharge.

3. If the discharge is uniform over its electrodes of 5 cm radius, estimate the average bulk ionisation rate per unit volume.

3.14 Answers

1. According to the Townsend criterion, the minimum gap for self-sustained discharge is given by

$$
\begin{aligned}
d &> \frac{1}{\alpha} \ln\left(\frac{1}{\gamma} + 1\right) \\
&= \frac{1}{1.1\ \mathrm{cm}^{-1}} \ln(51) \\
&= 3.6\ \mathrm{cm}.
\end{aligned}
$$

2. Using the same system as Question 1, the total discharge current is

$$
\begin{aligned}
I(d) &= \frac{I_{1e}(0)e^{\alpha d}}{1 - \gamma(e^{\alpha d} - 1)} \\
&= \frac{20 \times 10^{-12} \times e^{3\times1.1}\ [\mathrm{A}]}{1 - 0.02(e^{3\times1.1} - 1)} \\
&= 1.2\ \mathrm{nA}.
\end{aligned}
$$

3. The electron current at K is

$$
\begin{aligned}
I(K) &= \frac{I_{1e}(0)}{1 - \gamma(e^{\alpha d} - 1)} \\
&= \frac{20 \times 10^{-12}\ \mathrm{A}}{1 - 0.02(e^{3\times1.1} - 1)} \\
&= 44\ \mathrm{pA}.
\end{aligned}
$$

The difference in current between A and K is just less than 1.2 nA, hence

$$\begin{aligned}\frac{dN_e}{dt} &= \frac{I_A - I_K}{e} \\ &= \frac{1.2 \text{ nA}}{1.6 \times 10^{-19}\text{C}} \\ &= 0.75 \times 10^{10} \text{ s}^{-1}\end{aligned}$$

and

$$\frac{d\overline{n}_e}{dt} = \frac{\left(\frac{dN_e}{dt}\right)}{\pi r^2 l}.$$

Therefore

$$\begin{aligned}\frac{d\overline{n}_e}{dt} &= \frac{0.75 \times 10^{10}}{\pi(5 \times 10^{-12})^2 \times 3 \times 10^{-2}} \\ &= \frac{0.75 \times 10^{16}}{75\pi} \\ &= 3.2 \times 10^{13} \text{ m}^{-3}\,\text{s}^{-1}.\end{aligned}$$

Chapter 4

Fluid Model of a Plasma

A plasma may be described as an electrically-conducting fluid immersed in electric and magnetic fields. Starting from the foundations of hydrodynamics, i.e. the standard non-conducting fluid mechanics without electric and magnetic fields, that descibe the behaviour of atmospheres and oceans, conservation of mass leads to the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0,$$

where ρ is the mass density and $\mathbf{V}$ is the macroscopic fluid flow velocity. Further to this, conservation of momentum provides the equation

$$\rho \left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right) = -\nabla P + \mathbf{F},$$

where P is the isotropic fluid pressure and $\mathbf{F}$ represents external forces per unit volume. Finally, one more equation is necessary to determine the normal hydrodynamic fluid model: the equation of state. There are a number of models for this, such as

- Incompressible fluids:
$$\nabla \cdot \mathbf{V} = 0.$$
- Adiabatic fluids:
$$P\rho^{-\gamma} = \text{constant}.$$

- Isothermal fluids:

$$\frac{P}{\rho} = \text{constant}.$$

Whereas in ordinary hydrodynamics $\mathbf{F}$ would be a force, for example gravity, moving from ordinary hydrodynamics to the fluid model of a plasma, i.e. to hydromagnetics or magnetohydrodynamics (MHD), further to forces such as gravity one must also include the electric and magnetic forces given by Lorentz' force law;

$$\mathbf{F} = q\mathbf{E} + \mathbf{J} \times \mathbf{B},$$

where q is the charge density and $\mathbf{J}$ is the current density. We can then use two Maxwell equations;

$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt},$$

as well as the vector form of Ohm's law;

$$\mathbf{J} = \sigma\left(\mathbf{E} + \mathbf{V} \times \mathbf{B}\right),$$

where σ is the electrical conductivity. The final equation that allows the modelling of the fluid model of plasma is the conservation of electrical charge:

$$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

The plasma fluid models can be either single- or two-fluid. When using the single-fluid model of a plasma it is possible under some circumstances to simplify the set of MHD equations to provide a simpler set of so-called "ideal MHD" equations. These ideal MHD equations apply when the electrical conductivity of the plasma is high enough to be approximated to an infinite electrical conductivity. In summary, the set of ideal hydromagnetic or ideal magnetohydrodynamic equations is:

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) &= 0 \\ P\rho^{-\frac{5}{3}} &= \text{constant} \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\ \rho \frac{d\mathbf{V}}{dt} &= -\nabla P + \mathbf{J} \times \mathbf{B} \\ \nabla \times (\mathbf{V} \times \mathbf{B}) &= \frac{\partial \mathbf{B}}{\partial t}.\end{aligned}$$

4.1 Magnetic Pressure

Considering the ideal MHD equation

$$\rho \frac{d\mathbf{V}}{dt} = -\nabla P + \mathbf{J} \times \mathbf{B} \tag{4.1}$$

in the steady state, i.e. $\frac{d\mathbf{V}}{dt} = 0$, then

$$\nabla P = \mathbf{J} \times \mathbf{B}.$$

However, the steady state of Maxwell's equation

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt},$$

reverts to Ampère's law, i.e.

$$\mathbf{J} = \frac{\nabla \times \mathbf{B}}{\mu_0},$$

which is also one of the ideal MHD equations. Furthermore, substituting for $\mathbf{J}$ into Equation (4.1), one obtains

$$\begin{aligned}\nabla P &= \mathbf{J} \times \mathbf{B} \\ &= \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} \\ &= -\frac{1}{\mu_0} \mathbf{B} \times (\nabla \times \mathbf{B}).\end{aligned}$$

Expanding the vector triple product gives

$$\nabla P = -\frac{\nabla}{\mu_0}\left(\frac{B^2}{2}\right) + \frac{1}{\mu_0}\left(\mathbf{B}\cdot\nabla\right)\mathbf{B}. \tag{4.2}$$

One should note that the dyadic term $(\mathbf{B}\cdot\nabla)$ is equal to zero when $\mathbf{B}$ and ∇B are orthogonal, hence from Equation (4.2)

$$\nabla\left(P + \frac{B^2}{2\mu_0}\right) = 0,$$

where the first term is the plasma kinetic pressure and the second is the magnetic pressure. This shows that $P + \frac{B^2}{2\mu_0}$ is constant over space and hence this explains why the magnetic pressure of a magnetic field can confine the kinetic pressure of a plasma.

4.1.1 Magnetic Confinement

Where the magnetic field decreases to a minimum, the plasma kinetic pressure can increase to a maximum such that the spatial constancy of $P + \frac{B^2}{2\mu_0}$ is maintained. The parameter β, defined as the ratio of the thermodynamic pressure to the magnetic pressure, is used to classify the efficiency of the magnetic confinement, with both "local" and "global" β being used. Local β is a function of radius, r, and is defined by the ratio of the local kinetic pressure to the local magnetic pressure;

$$\beta(r) = \frac{P(r)}{\frac{B^2(r)}{2\mu_0}},$$

which can take values in the range $0 < \beta(r) < \infty$. Global β is a single value applied to the entire confinement system and is defined as

$$\begin{aligned}\beta(r) &= \frac{P(\text{at core of plasma, } r = 0)}{\frac{B^2}{2\mu_0}\ (\text{at outer edge of plasma})} \\ &= 1 - \frac{B^2(\text{at core of plasma, } r = 0)}{B^2\ (\text{at outer edge of plasma})},\end{aligned}$$

which can take values in the range $0 < \beta < 1$ and is sometimes expressed as a percentage, which $\beta = 1$ corresponding to 100%. The global β value can be regarded as a measure of the efficiency with which the magnetic field is used to confine a plasma in a particular confinement system.

In the low-β case the remaining magnetic field at $r = 0$ is effectively being "wasted" for confinement purposes and a far higher magnetic field is therefore required to confine the same plasma kinetic pressure. In the high-β case all of the magnetic field is efficiently used for confinement.

Example 4.1

Magnetic mirror traps are typically "low-β" devices whereas theta-pinches are usually "high-β" devices.

For efficiency the aim in magnetic confinement of plasmas is to obtain as high a β value as possible. In practice, high values of β can sometimes lead to plasma instabilities, therefore plasma confinement experiments in controlled thermonuclear fusion research usually try to obtain the highest β value consistent with achieving a stable plasma.

Example 4.2

In the high-β case all of the magnetic field is efficiently used for confinement. The high-β example, assuming a global $\beta = 1$, means that there is zero magnetic induction at $r = 0$ and so the plasma is confined with a maximally efficient use of the magnetic field and the minimum magnetic field of such is used.

4.2 The Pinch Effect

The pinching force is explained by parallel current filaments attracting each other, with "like currents" attracting and "like charges" repelling. If the current filaments are free to move as in a fluid then pinching will occur. The force on each current-carrying filament arises from integrating the force where $\mathbf{J}$ is the current density in the filament and $\mathbf{B}$ is the magnetic flux density created by the currents flowing the other filaments. The plasma cross-section is then pinched inwards until a pressure balance is reached where the plasma kinetic pressure and the magnetic pressure balance each other:

$$\nabla \left(P + \frac{B^2}{2\mu_0} \right) = 0.$$

The $\mathbf{J}_z \times \mathbf{B}_\theta$ force is radially inwards, resulting in a pinching of the plasma columns. It is an easy extrapolation to progress from a linear pinch to a toroidal punch by bending the column around into a torus. In the toroidal pinch case there are no electrodes and so inducing the current in the toroidal plasma is usually achieved by a transformer action.

The pinch effect occurs in plasma carrying a high enough current, including astrophysical plasmas and laboratory plasmas. The pinch effect is sometimes called the Bennett pinch. The pinch effect has been invoked in lightning discharges, solar flares, current sheets, Z-pinches, theta-pinches, toroidal punches, plasma focussing etc.

4.2.1 TOKAMAKs

Adding a strong toroidal field to a confined plasma results in the mechanism behind TOKAMAK. Developments and improvements of the TOKAMAK have controlled all of the major plasma instabilities, but to optimise energy output the ITER project plans to upscale the research to a gigwatt-size power plant.

TOKAMAK is a transliteration of the Russian word Токамак, which in itself is an acronym for "**то**роидальная **ка**мера в **ма**гнитных **к**атушках (pronounced *toroidal'naya kamera v magnitnykh katushkakh*)" - **to**roidal **cha**mber in **ma**gnetic **c**oils. These TOKAMAKs must be extremely large, requiring $nT\tau_E \geq 1 \times 10^{21}\ \mathrm{m}^{-3}\,\mathrm{keV\,s}^{-1}$, with n being the plasma density, T being the plasma temperature which is usually in the range 10-20 keV for net power production in D-T plasmas, and τ_E being the confinement time. It has been long known that a high density as well as a high plasma current, I_p - which in turn requires high magnetic fields, B_t - and a large size are required for stable energy production. Furthermore, some theories claim that the temperature gradient is limited so that only high temperatures are permitted inside large devices. Finally, prolonged energy confinement times require large chamber dimensions due to large bodies losing heat much slower than smaller bodies (e.g. babies cool much faster than fat people). Additionally, gigwatt size power station must be large otherwise the device would melt; magnets must be protected from neutrons, hence requiring > 1 m of shielding.

4.3 Problem

Calculate the magnitude of the minimum externally applied magnetic field, **B**, that is required to confine a plasma consisting entirely of protons and electrons, in which the electron number density is 8×10^2 m^{-3} and the electron and ion temperatures are each equal to 1.2×10^8 K. You may assume that the maximum value of β is used, i.e. that the magnitude of the magnetic field is zero at the centre of the plasma.

4.4 Answer

A common mistake is to forget that ions can also exert a pressure. The number densities and the temperatures of the ions and electrons are equal in this example. The pressues exerted by the ions and electrons are therefore equal. In this example all of the magnetic field is used to balance the kinetic pressure of the plasma; the magnetic field being zero at the centre of the plasma.

$$\begin{aligned} P = \frac{B^2}{2\mu_0} &= n_e k_B T_e + n_i k_B T_i \\ &= 2 n_e k_B T_e \\ \rightarrow B &= (4\mu_0 n_e k_B T_e)^{\frac{1}{2}} \\ &= 2.58 \text{ T}. \end{aligned}$$

Chapter 5

Fusion Plasmas

5.1 Why Fusion?

There is a great need for controlled thermonuclear fusion due to humanity's reliance upon fossil fuels of which there is only 40 years of fuel remaining and 30 years of ^{235}U for nuclear fission. Furthermore, renewable energies will never be capable of producing 100% of the energy required for current levels of population, ignoring population growth and greater energy consumption with technological advancements.

5.2 Physical Reason for Fusion Power

By fusing together light nuclei or by fissioning heavy nuclear, energy is released. There are a number of mechanisms to produce energy by fusion, the easiest to achieve being $\mathrm{D} + \mathrm{T} \rightarrow {}^4\mathrm{He} + \mathrm{n}$, but it requires the tritium to be bred from ^{6}Li: ${}^6\mathrm{Li} + \mathrm{n} \rightarrow {}^4\mathrm{He} + \mathrm{T}$. Other mechanisms include

- $\mathrm{D} + {}^3\mathrm{He} \rightarrow {}^4\mathrm{He} + \mathrm{p}$
- $\mathrm{D} + \mathrm{D} \rightarrow {}^3\mathrm{He} + \mathrm{n}$
- $\mathrm{D} + \mathrm{D} \rightarrow {}^3\mathrm{T} + \mathrm{p}$

The deuterium and tritium nuclei need to be in a plasma state because if the nuclei are made to collide in the form of a pair of opposing beams, scattering occurs more often than fusion reactions. In a plasma the nuclei can collide sufficiently often for fusion reactions to occur in some of the collisions and energy to be released. In a collision the Coulomb repulsive barrier needs to be overcome before the nuclei approach closely enough for a fusion reaction to occur. The collisions therefore need to be made with sufficient kinetic energy and thus the plasma temperature is required to be sufficiently high.

The deuterium-tritium fusion reaction rate increases rapidly with temperature until it maximises near 70 keV and then drops off gradually. The radiation loss from the plasma due to bremsstrahlung increases relatively slowly, as $T^{\frac{1}{2}}$. The rapidly increasing reaction rate will therefore overcome the bremsstrahlung radiation losses for a range of temperatures around $10 - 70$ keV. The peak cross section of collisions occurs at about 20 keV.

5.3 Lawson Criterion

First derived by John Lawson in 1957, the **Lawson criterion** is an important measure of a system that defines the conditions required for a fusion reactor to reach ignition and sustain itself, i.e. the heating of the plasma by the products of the fusion reactions is sufficient to maintain the temperature of the plasma against all losses without external power input. As originally formulated, the Lawson criterion gives a minimum required value for the product of the plasma (electron) density, n_e, and the **energy confinement time**, τ_E. Later analyses suggested that a more useful figure of merit is the "triple product" of density, confinement time, and a plasma temperature, T. The triple product also has a minimum required value, and the term "Lawson criterion" often refers to this inequality. For the deuterium-trition reaction, the physical value of the triple product is about

$$n_e T \tau_E > 3 \times 10^{21}\ \mathrm{m}^{-3}\,\mathrm{keV\,s}.$$

The plasma must be contained at a temperature of order 10^8 K where D-T fusion reactions have a sufficiently large cross section. The power produced per unit volume for a density n goes as n^2. Furthermore, thermal energy density goes as n and the rate of energy loss from the system goes as $\frac{n}{\tau}$, where τ is a characteristic time for energy loss (energy containment time). For the power production rate to be greater than the loss rate, $n\tau$ is required to be large enough - it is this that the Lawson criterion describes. Plasmas contained by a magnetic field are usually of density on order 10^{20} m^{-3} and τ is on the order of seconds.

5.4 Methods of Achieving Controlled Nuclear Fusion

The fusion reactor can sustain itself if enough of the energy produced goes into keeping the fuel hot.

5.4.1 Gravitational Confinement

One force capable of confining the fuel well enough to satisfy the Lawson criterion is gravity. The mass needed, however, is so great that gravitational confinement is only found in stars, e.g. the sun. Even if the more reactive fuel deuterium were used, a mass as great as the moon would be required.

5.4.2 Magnetic Confinement

Since plasmas are very good electrical conductors, magnetic fields can also confine fusion fuel. A variety of magnetic configurations can be used, e.g. toroidal confinment, especially TOKAMAKs and stellarators. It is for these magnetically confined plasma that auxiliary heating, i.e. high energy neutral beam injection and high power RF, are very often employed. The main magnetic field is in a toroidal direction which is produced by toroidal field coils. However, purely toroidal fields cannot confine the plasma, poloidal fields are also produced by inducing currents in the plasma. The current is induced by changing magnetic field though the centre of the torus, which also heats the plasma.

5.4.3 Inertial Confinement

A third confinement principle is to apply a rapid pulse of energy to a large part of the surface of a pellet of fusion fuel, causing it to simultaneously "implode" and heat to very high temperature and pressure. To achieve these extreme conditions, the initially cold fuel would be explosively compressed. Inertial confinement is attempted in "controlled" nuclear fusion, where the driver is a laser beam.

5.5 Charged Particles in a Magnetic Field

It has been shown that the relativistically correct cyclotron frequencies are given by

$$\omega = \frac{qB}{\gamma m},$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}.$$

If there are both perpendicular and parallel velocities then the motion of the particles is helical rather than circular in a plane. When $v_\perp = v\sin(\theta)$ and $v_\parallel = v\cos(\theta)$, the kinetic energy is given by

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}mv_\perp^2 + \frac{1}{2}mv_\parallel^2. \end{aligned}$$

When an electric field is added to the magnetic field, such that the fields are $\mathbf{B} = (0, 0, B)$ and $\mathbf{E} = (0, E_\perp, E_\parallel)$, a non-zero $E_\parallel$ gives rise to particle motion along the magnetic field and hence very large currents can flow. The usual assumption in single particle theory is that $E_\parallel = 0$. Considering the effect of finite $E_\perp$ combined with a magnetic field, $E_\perp$ and $\mathbf{B}$ combine to produce a drift velocity of the guiding centre:

$$\mathbf{v}_{E_\perp} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}.$$

This drift, which is sometimes called the "$\mathbf{E} \times \mathbf{B}$" drift, depends only upon the fields and not upon the signs of the charges and is dangerous for plasma containment as the whole plasma drifts off in the same direction.

Theorem 5.1: Guiding Centre Drift for Non-Electromagnetic Forces

Suppose that there is a non-electromagnetic force, $\mathbf{F}$, which is then equivalent to an electric field, $\mathbf{E}_F$, where

$$\mathbf{E}_F = \frac{\mathbf{F}}{q},$$

e.g. $\mathbf{F}$ could be the force of gravity. Comparing this with the formula for the $\mathbf{E} \times \mathbf{B}$ drift then the guiding centre drift for a non-electromagnetic force, $\mathbf{v}_f$, is given by

$$\mathbf{v}_f = \frac{\mathbf{F} \times \mathbf{B}}{qB^2}.$$

Note that this drift is in opposite directions for oppositely-charged particles.

If a gradient in perpendicular to a magnetic field exists within the field then a guiding centre drift occurs, resulting in oppositely-charged particles moving in opposite directions. This can result in a charge separation and therefore a buildup of electric field. The guiding centre drift caused by ∇B being perpendicular to $\mathbf{B}$ causes a charge separation, which then sets up an electric field perpendicular to $\mathbf{B}$, which in turn produces a disastrous $\mathbf{E} \times \mathbf{B}$ drift, sending all of the plasma particles radially outward toward the walls as the electric field that is generated is perpendicular to the magnetic field. The solution to this is to add a poloidal magnetic field.

Introducing a poloidal magnetic field, the $\mathbf{B}$ field lines now spiral as they transit the torus. The $\mathbf{E}$ field resulting from charge separation is then resolved by connection along the spiralling $\mathbf{B}$ field lines. There hence ceases to be a disastrous $\mathbf{E} \times \mathbf{B}$ drift and magnetic confinement is possible. However, plasma confinement is not perfect as particles and heat diffuse from the plasma centre toward the outside. The losses connected to this transport are considerable.

5.6 The Electrical Resistivity of a Plasma

The electrical conductivity of a plasma, σ, is given by

$$\sigma = \frac{6\pi\sqrt{3}}{\sqrt{m_e}}\left(\frac{\epsilon_0}{e}\right)^2 \frac{(k_B T_e)^{\frac{3}{2}}}{Z\ln(\Lambda)},$$

where

$$\Lambda = \frac{12\pi}{Z}\left(\frac{\epsilon_0 k_B T_e}{e^2}\right)^{\frac{3}{2}} \frac{1}{\sqrt{n_e}}.$$

One should note that the electrical conductivity is independent of n and depends upon $T_e^{\frac{3}{2}}$. It is more correct to say that the electrical conductivity is "insensitive" to n because the number density, n, is included in the expression for $\ln(\Lambda)$. Also, the plasma conducts electricity better as it gets hotter, which contrasts with metallic conductors like copper, where the electrical conductivity decreases with increasing temperature.

Recalling that the resistivity, η, is related to the electrical conductivity, σ, by the relationship $\eta = \frac{1}{\sigma}$, and using the the equation for σ, it is possible to write an equation for η:

$$\eta \approx \frac{190\ln(\Lambda)}{T_e^{\frac{3}{2}}}\ \Omega\,\mathrm{m},$$

where T_e is in degrees Kelvin and $Z = 1$ has been assumed, i.e. singly-charged positive ions. This is known as the "classical" or "Spitzer" resistivity, as Lyman Spitzer was the first to obtain this value for the electrical resistivity of a plasma in the 1950's.

The electrical resistivity of a plasma at a temperature of $T_e = 10^7$ K is comparable with that of copper at room temperature. This increase in electrical conductivity of a plasma as the temperature increases means that the electrical resistivity of a plasma drops with increasing temperature. This has important consequences for heating magnetically confined plasmas to thermonuclear reactor temperatures using Ohmic (I^2R) heating. By passing a current through a typical magnetically confined plasma, Ohmic heating is effective up to plasma temperatures of around 10^7 K, corresponding to about 1 keV, but for higher temperatures "auxiliary heating methods" are needed as the plasma resistivity drops below that of copper and therefore the transfer of energy to the plasma becomes inefficient. Such axuiliary heating methods include RF heating and high energy neutral beam injection.

5.7 Plasma Heating

Fusion requires ion thermal energies of around $10 - 20$ keV, for which a number of methods can be implemented.

5.7.1 Ohmic Heating

Ohmic heating uses a current driven around a plasma, with the resistivity producing heat. As resistivity goes as $T^{-\frac{3}{2}}$ this method becomes less efficient as the temperature increases. Initial heating is by the Joule effect associated with the current flowing in the plasma, necessary to create the TOKAMAK magnetic configuration. As the temperature increases further the collision frequency drops and the resistance falls and the Ohmic heating saturates at around 10^7 K. Around this temperature plasmas become better conductors than copper.

5.7.2 Neutral Beams

Neutral beam heating utilises a beam of high energy neutral particles into plasma, which are ionised by charge exchange or collisions with ions and electrons. Once ionised, the beam particles slow down by collision with the background plasma, leading to overall heating, however parameters are required to be adjusted so that the beam is neither absorbed nor pass straight through the plasma. Large machines like JET have tens of MW of neutral beam power.

5.7.3 Non-Inductive Current Drive

Driving the current with a varying magnetic field through the core cannot produce a steady state device, however in large machines the current could be driven for time of order 400 s. The loop voltage is only of order of a few volts, for a current of several megaampère. Both neutral beams and radio frequency waves can drive currents if sent in at an angle around the torus. This can augment the inductive current and also change the current profile, something which might be useful for the control of instabilities.

5.7.4 Radio Frequency Heating

Another method of non-inductive heating of plasmas is to use powerful electromagnetic waves. The general principle of this is to fire radio frequency waves of frequency ω into a plasma. The wave mode propagates into the plasma and is then absorbed by **Landau** or **cyclotron damping**.

Definition 5.1: Landau Damping

Landau damping is the decrease of plasma spatial charge wave amplitudes due to an irreversible energy exchange between an external electromagnetic radiation and the plasma particles forming the charge wave. If the electromagnetic waves and the charge waves propagate with a phase velocity v_{ph} then any particles that are slightly slower than v_{ph} will be accelerated by the electric field of the charge wave to move with phase velocity v_{ph}, whereas particle moving slightly faster will be decelerated to v_{ph}.

Definition 5.2: Cyclotron Damping

Collisional damping is ineffective in TOKAMAK conditions as it falls off as $T^{-\frac{3}{2}}$. Resonant particles - when the frequency of the electromagnetic wave matches that of the plasma - absorb energy, which is then dissipated to the rest of the plasma by collisions. Theoretical analysis requires looking at wave propagation and absorption, coupled with the **Fokker-Planck equation** to look at the effect of collisions and transfer of energy to the bulk. JET has tens of MW of ion cyclotron heating and ITER will have both ion and electron cyclotron systems.

Definition 5.3: Fokker-Planck Equation

The Fokker-Planck equation describes the time evolution of a probability distribution function that is subject to drag and random forces, such as Brownian motion.

Plasmas are subject to resonant absorption, for which there are two mechanisms to heat the plasma via radio frequency:

- **Cyclotron Absorption**
  The RF couples to the plasma at the rotation frequency of a species in their trajectory around the magnetic field lines.
- **Landau Absorption**
  The RF couples to the plasma such that the wave and particle have nearly the same velocity of propagation.

Some relevant equations for plasma frequencies include the following:

$$\text{Plasma frequency} \qquad \omega_{pe} = \left(\frac{n_e e^2}{\epsilon_0 m_e}\right)^{\frac{1}{2}}$$

$$\text{Ion/Electron cyclotron frequency} \qquad \omega_{ci,e} = \frac{eB}{\gamma m_{i,e}}$$

$$\text{Upper hybrid frequency} \qquad \omega_{UH}^2 = \omega_{ce}^2 + \omega_{pe}^2$$

$$\text{Lower hybrid frequency} \qquad \frac{1}{\omega_{LH}^2} = \frac{1}{\omega_{ci}^2 + \omega_{pi}^2} + \frac{1}{|\omega_{ci}\omega_{ce}|}$$

5.8 Heating at the Ion Cyclotron Frequency, f_{ci}

The ion cyclotron resonance heating (ICRH) system uses a wave propagating perpendicularly to the magnetic field surfaces at a frequency near to that of the gyration frequency of one of the ion populations, which for deuterium at 5 T is 38 MHz. The gyration frequency, $\omega = \frac{qB}{m}$, depends on the mass of the ion and the magnetic field whose intensity drops across the TOKAMAK, enabling localisation of the energy transfer region by simply adjusting the wave frequency.

Unfortunately, resonance cyclotron absorption is not possible on a plasma with a single ion component screening effect, one then resorts to a so-called **minority ion cyclotron heating** scenario, which consists of using a plasma with a majority of deuterium ions and a small percentage of hydrogen ions. The frequency on the hydrogen is then adjusted, which has a lower mass than that of deuterium, and the wave is to a great extent absorbed by the hydrogen ions, whose energy increases by several hundred electronvolts on each passage of their trajectory in the resonance zone. They then transmit their energy to electrons by collision, which in turn heats up the deuterium ions.

Amplifier chains generate the ICRH electromagnetic waves, each chain with a powerful (2 MW) tetrode tube in the final stage. Transmission lines that conduct ICRH waves from the generators to the JET TOKAMAK are low loss coaxial cables. The transmission lines terminate in 4 ICRH antennae that are installed within the JET inner wall and that are slotted in the front. Each antenna consists of four conductors (straps) and each strap is fed by a separate generator. The ICRH electromagnetic waves cannot propagate in the JET vessel vacuum (their wavelength being too long) so that the antenna must be as close to the plasma as possible.

5.9 Electron Cyclotron Heating (ITER)

Two modes of ECRH are possible; the O (**ordinary**) mode and the X (**extraordinary**) mode. As in the case of ion cyclotron heating, the interaction takes place when the electron crosses a resonance layer, determined by the electromagnetic frequency used, and depending on the magnetic field. The advantage of this type of heating is to produce very local deposits, and it can be used as a precision tool to take energy to a well-determined location in the plasma, which can have a stabilising effect on instabilities. In contrast to the two others, this mode of heating is less sensitive to edge conditions at the level of the highly simplified antenna, making coupling easier in a wider range of plasma parameters.

5.10 Thermal Conductivity

Just as the electrical conductivity/resistivity has been discussed, the thermal conductivity - another transport coefficient - can also be derived[1]. Thermal conductivity, κ, is given by

$$\kappa = \kappa_0 \frac{T_e^{\frac{5}{2}}}{\ln(\Lambda)},$$

[1]See Boyd and Sanderson for such a derivation.

where κ_0 is a constant, hence plasmas conduct heat very well at high temperatures. The diffusion, D, is therefore given by

$$D = (\text{const})\frac{T^{\frac{5}{2}}}{\ln(\Lambda)},$$

meaning that plasmas rapidly diffuse at high temperatures.

D-T fusion produces 14 MeV neutrons which escape the plasma and 3.5 MeV α particles. In a reactor the aim is to confine the latter until they have given up their energy and reached the temperature of the bulk of the plasma. With good enough confinement, ignition can be reached, i.e. a self-sustaining reaction with no need for external heating. After particles have lost energy one requires them to diffuse out of the plasma in order to avoid filling the chamber with a particle "ash". The neutrons will be absorbed in a lithium blanket - lithium produced by the fusion reaction - which then regenerates tritium, i.e. $^6\text{Li} + \text{n} \rightarrow {}^3\text{T} + {}^4\text{He}$.

5.11 JET and ITER

JET is currenty the largest TOKAMAK in operation, which has produced fusion power of 16 MW. ITER is in early stages of construction at Caderache in the South of France. Plasma operation in ITER is expected to start 2025 and the machine is expected to run for around 30 years. It will need to have superconducting coils and when executed it will produce 500 MW of power sustained for 400 seconds. ITER will explore the technologies around lithium blankets and will require multiple mechanisms for heating, including neutral beams, and ion and electron cyclotron resonances. ITER is essentially like JET but with the linear dimensions of the plasma chamber scaled up by a factor of 2. It will build on the success of JET which will be used to refine its design and operation. A recent JET upgrade improved upon the walls used in ITER; beryllium with tungsten divertor plates and carbon are now used, whereas this was previously unsuitable for reactors since it absorbs tritium.

5.11.1 L-H Transition

When a magnetically confined plasma is strongly heated and a threshold heating power is exceeded, the plasma may spontaneously transition from a **low confinement** (L-mode) state to a **high confinement** (H-mode) state. In 1990 it was found in the ASDEX TOKAMAK in Germany that with a strong neutral beam there is a transition to a higher confinement regime, with confinement time typically around twice that of a previous low confinement mode. This has now been seen on many TOKAMAKs and can also be produced by RF heating. The transition is associated with a rapid increase in edge density as a result of a decrease of tranport at the edge. This then produces an increase in density throughout the plasma - essentially shifting the internal profile up from an edge pedestal.

Definition 5.4: High Confinement

The high confinement mode is a highly stable confinement of plasmas, allowing for increased confinement times.

5.11.2 Confinement: H-Mode

The high confinement mode is standard ("baseline") operation for ITER, approximately doubling the confinement time, τ_E. The standard H-mode will have **edge-localised modes** (ELMs).

Definition 5.5: Edge-Localised Modes

Edge-localised modes (ELMs) are associated with the H-mode and are characterised by ejection of the plasma from the steep density gradient at the edge. ELMs provide density and impurity control, but are potentitally damaging if not controlled. Understanding and controlling these is a major current field of research since they can place severe heat loads on the vacuum chamber. The energy loss per edge eruption in ITER is predicted to be greater than the material damage limit, hence ITER is planning to use in-vessel coils to control ELMs.

The design objective of ITER is to achieve $Q = 10$, where Q is given by

$$Q = \frac{\text{Fusion power}}{\text{Heating power}}.$$

The transition to H-mode can be routinely reproduced in modern TOKAMAKs. The existence of this transport barrier at the edge is generally attributed to shear flow which break up turbulent eddies and so inhibits transport. Under some conditions it is also possible to produce internal transport barriers. Empirical H-mode scaling can be used to preduct the confinement properties of ITER.

5.11.3 Instabilities

The plasma in a TOKAMAK is far from equilibrium and is subject to a wide variety of instabilities. Complete suppression of these is impossible and the objective is to avoid those which will produce catastrophic loss of plasma. There are several main categories of instability:

- **Microinstabilities**
  MHD instabilities produce bulk motion of the plasma. Locally, particle distributions may be away from equilibrium, typically because of drift motions produced by gradients. This can produce small-scale instabilities which must be investigated using kinetic theory via tha Vlasov equation, both of which are later discussed. These do not generally have catastrophic effects, but contribute to anomalous transport.
- **Fluid instabilities**
  These instabilities cannot be altogether avoided, but their seriousness largely depends on the variation of the safety factor across the plasma. They also produce a β limit.
- **Sawtooth oscillations**
  These are when a rise in density and temperature are followed by a sudden fall, with this process repeating.

The instabilities that arise within a model depend on which model is used;

- **Ideal MHD**
  Fluid treatment with no resistivity

- **Resistive MHD**
  With no resistivity, magnetic field lines are frozen into the plasma so no change in field topology is allowed. With resistivity reconnection is allowed - this allows formation of magnetic islands which are produced by a resistive tearing mode.

5.11.4 Key Figures

Parameter	JET	ITER
Fusion Power	< 16 MW	500 MW
Plasma Lifetime	$5 - 30$ s	400 s
Plasma Current	$3.2 - 4.8$ MA	15 MA
Auxiliary Heating	23 MW(NB), 15 MW(RF)	100 MW(NB, E&I-CRH, LHRF)
Major Radius	2.96 m	6.2 m
Minor Radius	$1.25 - 2.10$ m	2.00 m
Plasma Volume	100 m^3	840 m^3

To achieve these, one must confine in H-mode to achieve the $Q = 10$ design goal of ITER. Furthermore, the plasma must avoid major instabilities which could lead to complete loss of confinement. There are limits on how much power can be supplied to the surface materials.

5.11.5 ITER Baseline Scenario

The basline scenario for operating ITER and achieving $Q = 10$ is ELM H-mode:

- Power threshold $P_{LH} = 55$ MW, marginal with existing heating. Lower densities reduce the threshold up to a point.
- Once in H-mode, fusion power provides more than enough energy to stay in H-mode.
- The core temperature strongly depends on edge temperatures, so a huge effort is being put into understanding and predicting properties.

5.11.6 Power Handling

The handling of power has major implications for plasma operation because the energy must leave the core plasma to something else. This energy will come from both heating input and heating power. For instance, given 50 MW of heating power and $\frac{1}{5}$ of the 500 MW fusion power, the power output to divertors on ITER is around 150 MW. Intensive research activity on divertors to withstand such power include tilting the target plates to spread the power over a larger area and magnetic geometries like Super-X to reduce the power reaching the divertor targets.

Chapter 6

Two-Fluid Model

6.1 Waves in Unmagnetised Plasmas

Plasma ions and electrons interact via long-range electric and magnetic forces, leading to collective behaviours. There are three basic wave-modes that exist in unmagnetised plasmas:

- **Langmuir waves**
  Oscillations of plasma electrons are modified by thermal effects
- **Ion acoustic waves**
  Acoustic waves driven by electron and ion pressures being balanced by the ion inertia
- **Electromagnetic waves**
  Electromagnetic waves can be modified by the free plasma electrons

6.2 Basics of the Two-Fluid Model

The two-fluid model describes the plasma in terms of two distinct but intermixed fluids: ions and electrons. The ion and electron number densities, n_i and n_e, are obtained from the continuity equations;

$$\frac{\partial n_{i,e}}{\partial t} + \nabla \cdot (n_{i,e}\mathbf{v}_{i,e}) = 0.$$

The ion and electron fluid velocities, $\mathbf{v}_i$ and $\mathbf{v}_e$, are obtained from the momentum equations;

$$m_{i,e}\left(\frac{\partial \mathbf{v}_{i,e}}{\partial t} + (\mathbf{v}_{i,e}\cdot\nabla)\mathbf{v}_{i,e}\right) = Zq(\mathbf{E} + \mathbf{v}_{i,e}\times\mathbf{B}) - \frac{\nabla P_{i,e}}{n_{i,e}},$$

where Zq is the charge state of the ions ($= Ze$) and electrons ($-e$), and $P_{i,e}$ are the ion and electron thermal pressures.

Recall Maxwell's equations as well as some derivatives:

Faraday's law	$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}$
Ampère-Maxwell's law	$\nabla\times\mathbf{B} = \mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$
Gauss' law	$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$
No monopoles	$\nabla\cdot\mathbf{B} = 0$
Electric current density	$\mathbf{j} = Zen_i\mathbf{v}_i - en_e\mathbf{v}_e$
Electric charge density	$\rho = Zen_i - en_e.$

Henceforth it will be assumed that plasma atoms are singly-ionised, i.e. the ion charge is given by $Z = 1$.

6.3 Electrostatic Waves

The ion acoustic and Langmuir waves are **electrostatic waves**. The electric field, $\mathbf{E} = -\nabla\phi$, is expressed as the gradient of the **electrostatic potential**, ϕ. There are no magnetic fluctuations since it follows from Faraday's law that

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} = -\nabla\times\nabla\phi = 0.$$

From Gauss' law,

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} = \frac{e}{\epsilon_0}(n_i - n_e),$$

one then obtains Poisson's equation, i.e.

$$-\nabla^2\phi = \frac{e}{\epsilon_0}(n_i - n_e). \tag{6.1}$$

The Poisson equation is coupled with the ion and electron continuity equations as well as the momentum equations, namely

$$m_{i,e}\left(\frac{\partial \mathbf{v}_{i,e}}{\partial t}+(\mathbf{v}_{i,e}\cdot\nabla)\mathbf{v}_{i,e}\right)=q\mathbf{E}-\frac{\nabla P_{i,e}}{n_{i,e}}=-q\nabla\phi-\frac{\nabla P_{i,e}}{n_{i,e}}.$$

If the potential, ϕ, was equal to zero then the acoustic equations for the ions and electrons uncouple. However, any changes in the electron or ion densities give rise to a response in the potential due to Equation (6.1). Therefore the ion and electron densities are strongly tied together.

Example 6.1: Gravitational vs Electrostatic Forces

For the purpose of a gedanken experiment, consider 1 kg of nucleons and 1 kg of electrons a distance, d, 1 m apart. What would be the relative forces between the two bodies?

The gravitational force, F_g, between the bodies would be given by

$$F_g=\frac{Gm_N m_e}{d^2},$$

whereas the electrostatic force between the bodies would be

$$F_e=\frac{1}{4\pi\epsilon_0}\frac{Q_N Q_e}{d^2}.$$

Hence the ratio of these forces is given by

$$\frac{F_g}{F_e}=G4\pi\epsilon_0\cdot\frac{m_N m_e}{Q_N Q_e}.$$

6.3.1 Small-Amplitude Waves

A simplifying assumption is that the waves have small-amplitudes which oscillate around an equilibrium solution of the equations. The simplest equilibrium solution is that the plasma is at rest with equal electron and ion densities, i.e. $n_{i,e}=n_0=$ constant, together with $\phi=0, \mathbf{v}_{i,e}=0, \nabla P_{i,e}=0$.

6.3.2 Procedure of Linearisation

The linearisation procedure considers a perturbation from equilibrium such that $n_{i,e} = n_0 + n_{i1,e1}$ where n_0 is the equilibrium plasma density and $n_{i1,e1}$ are perturbation densities where $\|n_{i1,e1}\| \ll n_0$, $\phi = \phi_1$, and $\mathbf{v}_{i,e} = \mathbf{v}_{i,e1}$, where quadratic terms such as $n_{i1}\mathbf{v}_{i1}$ and $(\mathbf{v}_{i1} \cdot \nabla)\mathbf{v}_{i1}$ have been neglected. Furthermore, polytropic ion and electron pressures are assumed, i.e.

$$P_{i,e} = P_{i,e0}\left(\frac{n_{i,e}}{n_0}\right)^{\gamma_{i,e}},$$

where $P_{i,e0} = n_0 k_B T_{i,e0}$ are the equilibrium ion and electron pressures, and $\gamma_{i,e}$ are the ratios of specific heats (adiabatic constants) for the ions and electrons.

Taylor expanding the pressures about $n_{i,e} = n_0$ gives

$$P_{i,e} = P_{i0,e0} + P_{i1,e1},$$
$$\text{with } P_{i1,e1} = \gamma_{i,e} k_B T_{i0,e0} n_{i1,e1}.$$

The linearised equations are thus given by

Poisson's equation	$-\nabla^2\phi_1 = \frac{e}{\epsilon_0}(n_{i1} - n_{e1})$
Ion/Electron continuity	$\frac{\partial n_{i1,e1}}{\partial t} + n_0 \nabla \cdot \mathbf{v}_{i1,e1} = 0$
Ion/Electron momentum	$m_{i,e}\frac{\partial \mathbf{v}_{i1,e1}}{\partial t} = -q\nabla\phi_1 - \gamma_{i,e} k_B T_{i0,e0}\frac{\nabla n_{i1,e1}}{n_0}$

6.3.3 Langmuir Waves

Ions and electrons oscillate with opposite phases to one another, giving rise to a strong response in the potential and to high-frequency electron oscillations close to the plasma frequency. It is a good approximation to assume that the much heavier ions do not have time to move on this timescale, known as the cold ion limit. Then, using the lack of an ionic perturbation, i.e. $n_{i1} = 0$, and $\mathbf{v}_{i1} = 0$, the time derivative of the equation for electron continuity, the equation for the electron momentum, and using Poisson's equation to eliminate $\mathbf{v}_{e1}$ and ϕ_1, one gets

$$\begin{aligned}\frac{\partial^2 n_{e1}}{\partial t^2} &= -n_0 \nabla \cdot \frac{\partial \mathbf{v}_{e1}}{\partial t} \\ &= -\frac{n_0}{m_e}\left(e\nabla^2\phi_1 - \gamma_e k_B T_{e0}\frac{\nabla^2 n_{e1}}{n_0}\right) \\ &= -\frac{n_0 e^2}{m_e \epsilon_0} n_{e1} + \gamma_e \frac{k_B T_{e0}}{m_e}\nabla^2 n_{e1}.\end{aligned}$$

This can be written

$$\frac{\partial^2 n_{e1}}{\partial t^2} + \omega_{pe}^2 n_{e1} - \gamma_e v_{Te}^2 \nabla^2 n_{e1} = 0,$$

where $\omega = \sqrt{\frac{n_0 e^2}{\epsilon_0 m_e}}$ is the electron plasma frequency and $v_{Te} = \sqrt{\frac{k_B T_e}{m_e}}$ is the electron thermal speed.

For Langmuir waves it is a good approximation to use $\gamma_e = 3$. In the language of thermodynamics, the electrons perform one-dimensional adiabatic compression with one translational degree of freedom, $N = 1$, giving $\gamma_e = \frac{2+N}{N} = 3$. This value is also derived from kinetic theory using the Vlasov equation for waves with phase velocities much greater than the electron thermal speed. One can now assume that n_{e1} can be represented as a sum of plane waves proportional to $\exp(-i\omega t + i\mathbf{k}\cdot\mathbf{x})$ with different values of wave frequency ω and wave vector components k_x, k_y, k_z. The frequencies and wave vectors of each wave must then obey the dispersion relation with $\gamma_e = 3$, giving the Bohm-Gross relation;

$$\omega^2 = \omega_{pe}^2 + 3v_{Te}^2 k^2, \tag{6.2}$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$.

The lowest frequency for Langmuir waves occurs when the wavelength tends to infinite, giving rise to the so-called **cutoff frequency**, $\omega = \omega_{pe}$. One sees from the dispersion relation that the phase speed,

$$v_{ph} = \frac{\omega}{k} = \frac{\sqrt{\omega_{pe}^2 + 3v_{Te}^2 k^2}}{k},$$

decreases for large k, down to a minimum value $\sqrt{3}v_{Te}$. However, when the phase speed becomes comparable with the electron thermal speed, there are strong interactions between the wave and single electrons, which gives rise to Landau damping of the wave, where the wave energy goes into acceleration of electrons. This damping can only be described with kinetic theory (covered later) and is not included in the fluid model. As a rule of thumb, Langmuir waves are weakly Landau damped only when $k < 0.15\frac{\omega_{pe}}{v_{Te}}$. For $k > 0.25\frac{\omega_{pe}}{v_{Te}}$ the Langmuir waves are strongly Landau damped and the fluid model breaks down. Hence, the frequency of Langmuir waves are often very close to the plasma frequency. This can be used as a diagnostic to measure the electron number density in a plasma.

6.3.4 Ion Acoustic Waves

Ion acoustic waves arise when the plasma electrons and ions oscillate with the same phase. Since the ions are much heavier than the electrons, the electrons tend to follow the ions and neutralise the plasma. These oscillations have much lower frequencies than the Langmuir waves. A good approximation is to assume that the electrons have zero inertia. In this case, one can set $m_e = 0$ in the left-hand side of the electron momentum equation and integrating it;

$$0 = e\phi_1 - \gamma_e k_B T_{e0} \frac{n_{e1}}{n_0}.$$

For this case it is a good approximation to use $\gamma_e = 1$, corresponding to isothermal compression. This value can also be derived from kinetic theory for a wave whose phase velocity is much lower than the electron thermal speed. Solving for n_{e1} and inserting the result with $\gamma_e = 1$ into Poisson's equation and rearranging gives

$$\phi_1 - \lambda_{De}^2 \nabla^2 \phi_1 = \frac{k_B T_{e0}}{e} \frac{n_{i1}}{n_0}, \tag{6.3}$$

where $\lambda_{De} = \sqrt{\frac{k_B T_{e0} \epsilon_0}{n_0 e^2}}$ is the electron Debye length. A very useful relation for the Debye length and thermal velocity is

$$v_{Te} = \lambda_{De} \omega_{pe}.$$

In a similar procedure as for Langmuir waves, we now twice take the derivative of Equation (6.3) and use the equations for ion continuity and momentum to eliminate n_{i1} and $\mathbf{v}_{i1}$, resulting in

$$\frac{\partial^2}{\partial t^2}(\phi_1 - \lambda_{De}^2 \nabla^2 \phi_1) = c_s^2 \nabla^2 \phi_1 - \gamma_i v_{Ti}^2 \lambda_{De}^2 \nabla^4 \phi,$$

where $c_s = \sqrt{\frac{k_B(T_e+\gamma_i T_i)}{m_i}}$ is the ion acoustic speed and $v_{Ti} = \sqrt{\frac{k_B T_i}{m_i}}$ and is the ion thermal speed. An exponential ansatz can be made for the perturbed potential, i.e. $\phi_1 \propto \exp(-i\omega t + i\mathbf{k} \cdot \mathbf{x})$, giving the dispersion relation

$$\omega^2(1 + \lambda_{De}^2 k^2) = c_s^2 k^2 + \gamma_i v_{Ti}^2 \lambda_{De}^2 k^4.$$

The ion adiabatic constant $\gamma_i = 3$ is usually used, corresponding to one-dimensional adiabatic compression. It can also be derived from kinetic theory for wave phase speeds much higher than the ion thermal speed, $\frac{\omega}{k} \gg v_{Ti}$. Ion acoustic waves are weakly Landau damped only when $T_e \gg T_i$, in practice this is more like $T_e > 10T_i$. For $T_e < 3T_i$ the ion acoustic waves are strongly Landau damped.

Some useful limits to remember include:

- Long wavelength limit, $\lambda_{De}^2 k^2 \ll 1$, corresponding to $n_{e1} = n_{i1}$;

$$\omega^2 = c_s^2 k^2.$$

- Cold ion limit, $T_i = 0$ and $v_{Ti} = 0$;

$$\omega^2 = \frac{c_s^2 k^2}{1 + \lambda_{De}^2 k^2},$$

$$\text{with } c_s = \sqrt{\frac{k_B T_e}{m_i}}.$$

This has resonance for $\lambda_{De}^2 k^2 \gg 1$ at the ion frequency;

$$\omega = \frac{c_s}{\lambda_{De}} = \sqrt{\frac{e^2 n_0}{\epsilon_0 m_i}} = \omega_{pi}.$$

6.3.5 Electromagnetic Waves

The electromagnetic waves in the two-fluid model are transverse waves, i.e. the electric and magnetic fields are perpendicular to the direction of propagation. These fields are not generally associated with fluctuations in electron or ion density, hence in an homogeneous plasma one can take $n_{i,e} = n_0$, giving $\nabla\cdot\mathbf{v}_{i,e} = 0$. For such a situation, Gauss' law gives $\nabla\cdot\mathbf{E} = \frac{e}{\epsilon_0}(n_i - n_e) = 0$. One can also set the pressures to constancy, i.e. $\nabla P_{i,e} = 0$, hence the linearised momentum equations then read

$$m_{i,e}\frac{\partial \mathbf{v}_{i1,e1}}{\partial t} = q\mathbf{E}_1,$$

which are coupled with the Faraday and Ampère-Maxwell laws;

$$\frac{\partial \mathbf{B}_1}{\partial t} = -\nabla\times\mathbf{E}_1$$
$$\nabla\times\mathbf{B}_1 = \mu_0 n_0(\mathbf{v}_{i1} - \mathbf{v}_{e1}) + \frac{1}{c^2}\frac{\partial \mathbf{E}_1}{\partial t},$$

together with $\nabla\cdot\mathbf{B} = 0$. Taking the time derivative of the Ampère-Maxwell equation and using Faraday's law and lineared momentum equations to eliminate the time derivative of $\mathbf{v}_{i,e}$ and $\mathbf{B}_1$ then gives

$$-\nabla\times(\nabla\times\mathbf{E}_1) = \mu_0 n_0\left(\frac{1}{m_i} + \frac{1}{m_e}\right)e\mathbf{E}_1 + \frac{1}{c^2}\frac{\partial^2\mathbf{E}_1}{\partial t^2}.$$

Since $m_i \gg m_e$, one can usually neglect the term $\frac{1}{m_i}$ compared to $\frac{1}{m_e}$. Using vector calculus identities, the linear wave equation is given by

$$\frac{\partial^2\mathbf{E}_1}{\partial t^2} - c^2\nabla^2\mathbf{E}_1 = -\omega_{pe}^2\mathbf{E}_1,$$

where $\omega_{pe} = \sqrt{\frac{n_0 e^2}{\epsilon_0 m_e}}$ is the electron plasma frequency. The dispersion relation in this case is

$$\omega^2 = c^2k^2 + \omega_{pe}^2.$$

As earlier noted, the electromagnetic wave has a cutoff frequency $\omega = \omega_{pe}$ and electromagnetic waves with lower frequencies than this cannot propagate into the plasma. The electromagnetic wave (light) can be reflected by an **overdense** ($\omega_{pe} > \omega$) plasma. There is no Landau damping, since the phase speed, $\frac{\omega}{k}$, is always greater than the speed of light and hence cannot interact resonantly with electrons. The **group speed**, $\frac{\partial\omega}{\partial k}$, is smaller than the speed of light.

6.4 Waves in Magnetised Plasmas

A background magnetic field breaks the symmetry and makes the plasma **anisotropic**, i.e. the waves have different behaviours if they propagate along or across magnetic field lines. Electromagnetic waves with different polarisation have different propagation characteristics, and hence the plasma becomes **doubly refractive** for electromagnetic waves. These effects are caused by ions and electrons gyrating within a magnetic field and the plasma performing $\mathbf{E}\times\mathbf{B}$ drifts in low frequency electric fields perpendicular to the magnetic field. This modifies the behaviour of existing waves and introduces a number of new wave modes compared to an unmagnetised plasma.

6.4.1 Linearisation

In the two-fluid model the linearisation procedure assumes that $n_{i,e} = n_0 =$ constant and $\mathbf{B} = \mathbf{B}_0 =$ constant, together with $\mathbf{E} = 0$, $\mathbf{v}_{i,e} = 0$ and $\nabla P_{i,e} = 0$. For simplicity and without loss of generality, it is assumed that the magnetic field is directed along the z-axis, such that $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$.

The equilibrium is perturbed such that $n_{i,e} = n_0 + n_{i,e1}$, where $n_{i1,e1} \ll n_0$, and $\mathbf{B} = \mathbf{B_0} + \mathbf{B_1}$, where $|\mathbf{B}_1| \ll |\mathbf{B}_0|$, $\mathbf{E} = \mathbf{E}_1$ and $\mathbf{v}_{i,e} = \mathbf{v}_{i,e1}$. As before, polytropic ion and electron pressures are assumed, which can be Taylor expandd about $n_{i,e} = n_0$ and $P_{i,e} = P_{i,e0} + P_{i,e1}$, with $P_{i,e1} = \gamma_e k_B T_{i,e0} n_{i,e1}$. The ion and electron number densities are obtained from the continuity equations;

$$\frac{\partial n_{i1,e1}}{\partial t} + n_0 \nabla \cdot \mathbf{v}_{i1,e1} = 0,$$

whereas the ion and electron fluid velocities are obtained from the momentum equations;

$$m_{i,e}\frac{\partial \mathbf{v}_{i1,e1}}{\partial t} = q(\mathbf{E}_1 + \mathbf{v}_{i1,e1} \times \hat{\mathbf{z}} B_0) - \gamma_{i,e} k_B T_{i0,e0} \frac{\nabla n_{i1,e1}}{n_0}.$$

By dividing the ion and electron momentum equations by the repsective masses, one obtains

$$\frac{\partial \mathbf{v}_{i1,e1}}{\partial t} = \frac{q\mathbf{E}_1}{m_{i,e}} + \frac{q}{e}\mathbf{v}_{i1,e1} \times \hat{\mathbf{z}}\omega_{ci,ce} - \gamma_i v_{Ti,Te}^2 \frac{\nabla n_{i1,e1}}{n_0},$$

where $\omega_{ci,ce} = \frac{eB_0}{m_{i,e}}$ are the ion and electron cyclotron frequencies and $v_{Ti,Te} = \sqrt{\frac{k_B T_{i,e}}{m_{i,e}}}$ are the ion and electron thermal speeds. It is hence clear that gyromotion of the ions and electrons influence the behaviour of the plasma. One should also note that $\omega_{ci} = \omega_{ce}\frac{m_e}{m_i}$ whereas $\omega_{pi} = \omega_{pe}\sqrt{\frac{m_e}{m_i}}$, hence in general $\omega_{ci} \ll \omega_{pi}$. The linearised Maxwell's equations are

Faraday's law	$\frac{\partial \mathbf{B}_1}{\partial t} = -\nabla \times \mathbf{E}_1$
Ampère-Maxwell's law	$\nabla \times \mathbf{B}_1 = \mu_0 e n_0(\mathbf{v}_{i1} - \mathbf{v}_{e1}) + \frac{1}{c^2}\frac{\partial \mathbf{E}_1}{\partial t}$
Gauss' law	$\nabla \cdot \mathbf{E}_1 = \frac{e}{\epsilon_0}(n_{i1} - n_{e1})$
No monopoles	$\nabla \cdot \mathbf{B}_1 = 0$

Making the electrostatic assumption; $\mathbf{E}_1 = -\nabla\phi_1$, Gauss' law can be used to obtain Poisson's equation;

$$-\nabla^2\phi_1 = \frac{e}{\epsilon_0}(n_{i1} - n_{e1}),$$

which is coupled with the ion and electron momentum and continuity equations;

$$\frac{\partial \mathbf{v}_{i1,e1}}{\partial t} = -\frac{q}{m_{i,e}}\nabla\phi_1 + \frac{q}{e}\nabla_{i1,e1} \times \hat{\mathbf{z}}\omega_{ci,ce} - \gamma_{i,e} v_{Ti,Te}^2 \frac{\nabla n_{i1,e1}}{n_0};$$
$$\frac{\partial n_{i,e1}}{\partial t} + n_0 \nabla \cdot \mathbf{v}_{i,e1} = 0.$$

Plane wave solutions can now be ansantzed, i.e. $\phi_1, n_{i,e1}, \mathbf{v}_{i,e1}$ are proportional to $\exp(-i\omega t + i\mathbf{k}\cdot\mathbf{x})$, giving the **algebraic system of equations**:

$$\begin{aligned} k^2\phi_1 &= \frac{e}{\epsilon_0}(n_{i1} - n_{e1}); \\ -i\omega \mathbf{v}_{i1,e1} &= -\frac{q}{m_{i,e}} i\mathbf{k}\phi_1 + \frac{q}{e}\mathbf{v}_{i1,e1} \times \hat{\mathbf{z}}\omega_{ci,ce} - i\mathbf{k}\gamma_i v_{Ti,Te}^2 \frac{n_{i1,e1}}{n_0}; \\ -i\omega n_{i,e1} &+ n_0 i\mathbf{k}\cdot\mathbf{v}_{i,e1} = 0. \end{aligned} \quad (6.4)$$

For electrostatic waves, the electric field is parallel to the wavevector since $\mathbf{E}_1 = -i\mathbf{k}\phi_1$.

6.4.2 Parallel Propagation

Parallel propagation, where the wavevector is parallel to the magnetic field, i.e. $\mathbf{k} = k_z\hat{\mathbf{z}}$, will be considered. In such a case, Poisson's equation then yields

$$k_z^2\phi_1 = \frac{e}{\epsilon_0}(n_{i1} - n_{e1}),$$

while the continuity equations become

$$-i\omega n_{i1,e1} + n_0 i k_z v_{iz1,ez1} = 0.$$

The z-components of the velocities are obtained from Equation (6.4) as

$$-i\omega v_{iz1,ez1} = -\frac{q}{m_{i,e}} i k_z \phi_1 - i k_z \gamma_{i,e} v_{Ti,Te}^2 \frac{n_{i1,e1}}{n_0}.$$

One should note that the magnetic field has disappeared from the equations, i.e. electrostatic waves propagating along the magnetic field are not affected by the magnetic field - the reason for this is that $\mathbf{v}_{i,e1}$ are parallel to $\mathbf{B}_0$. In appropriate limits the Langmuir and ion acoustic waves are retained.

6.4.3 Perpendicular Propagation

For electrostatic waves propagating perpendicularly to the magnetic field, the electric field is perpendicular to the magnetic field. Due to this, the electron and ion fluid velocities are also perpendicular to the magnetic field. This then gives

$$k_\perp^2\phi_1 = \frac{e}{\epsilon_0}(n_{i1} - n_{e1});$$
$$-i\omega \mathbf{v}_{i,e\perp 1} = -\frac{q}{m_e} i\mathbf{k}_\perp \phi_1 + \mathbf{v}_{i,e\perp 1} \times \hat{\mathbf{z}}\omega_{ci,ce} - i\mathbf{k}_\perp \gamma_{i,e} v_{Ti,Te}^2 \frac{n_{i1,e1}}{n_0};$$
$$-i\omega n_{i1,e1} + n_0 i\mathbf{k}_\perp \cdot \mathbf{v}_{i,e\perp 1} = 0,$$

where $\mathbf{k}_\perp = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}}$ and $\mathbf{v}_{i,e\perp 1} = v_{ix,ex}\hat{\mathbf{x}} + v_{iy,ey}\hat{\mathbf{y}}$. Assuming that the first high-frequency waves are similar to Langmuir waves, where the ions can be taken to be stationary, the system of equations then reduces to

$$
\begin{aligned}
k_\perp^2 \phi_1 &= -\frac{e}{\epsilon_0} n_{e1} \\
-i\omega \mathbf{v}_{e\perp 1} &= \frac{e}{m_e} i\mathbf{k}_\perp \phi_1 - \mathbf{v}_{e\perp 1} \times \hat{\mathbf{z}}\omega_{ce} - i\mathbf{k}_\perp \gamma_e v_{Te}^2 \frac{n_{e1}}{n_0} \\
-i\omega n_{i,e1} &+ n_0 i\mathbf{k}_\perp \cdot \mathbf{v}_{i,e\perp 1} = 0.
\end{aligned}
$$

To obtain a dispersion relation for the wave, the strategy is to express the perturbation density n_{e1} in terms of ϕ_1 and then insert the result into the first of this system of equations. Writing out the x- and y-components of the second equation, one gets

$$
\begin{aligned}
-i\omega v_{ex1} &= \frac{e}{m_e} ik_x \phi_1 - v_{ey1}\omega_{ce} - ik_x \gamma_e v_{Te}^2 \frac{n_{e1}}{n_0} \\
-i\omega v_{ey1} &= \frac{e}{m_e} ik_y \phi_1 + v_{ex1}\omega_{ce} - ik_y \gamma_e v_{Te}^2 \frac{n_{e1}}{n_0}.
\end{aligned}
$$

Solving for $v_{ex1,ey1}$ gives

$$
\begin{aligned}
v_{ex1} &= \left(-\frac{e}{m_e}\phi_1 + \gamma_e v_{Te}^2 \frac{n_{e1}}{n_0} \right) \frac{k_x \omega - ik_y \omega_{ce}}{\omega^2 - \omega_{ce}^2} \\
v_{ey1} &= \left(-\frac{e}{m_e}\phi_1 + \gamma_e v_{Te}^2 \frac{n_{e1}}{n_0} \right) \frac{k_y \omega + ik_x \omega_{ce}}{\omega^2 - \omega_{ce}^2}.
\end{aligned}
$$

When inserted into the continuity equation these give

$$
n_{e1} = -\frac{n_0 e k_\perp^2 \phi_1}{m_e(\omega^2 - \omega_{ce}^2 - \gamma_e v_{Te}^2 k_\perp^2)}.
$$

Finally, by Poisson's equation one has

$$
k_\perp^2 \phi_1 = \frac{\omega_{pe}^2 k_\perp^2 \phi_1}{\omega^2 - \omega_{ce}^2 - \gamma_e v_{Te}^2 k_\perp^2}
$$

and the dispersion relation

$$
1 = \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2 - \gamma_e v_{Te}^2 k_\perp^2}. \tag{6.5}
$$

Equation (6.5) can be written

$$
\begin{aligned}
\omega^2 &= \omega_{pe}^2 + \omega_{ce}^2 + \gamma_e v_{Te}^2 k_\perp^2 \\
&= \omega_{UH}^2 + \gamma_e v_{Te}^2 k_\perp^2,
\end{aligned}
$$

where $\omega_{UH} \equiv \sqrt{\omega_{pe}^2 + \omega_{ce}^2}$ is the upper hybrid frequency. Here $\gamma_e = 3$ will be assumed. Fluid theory is only valid in weak magnetic fields such that $\omega_{pe}^2 \gg \omega_{ce}^2$, i.e. $\omega_{UH} \approx \omega_{pe}$. An exact expression for the dispersion relation obtained from kinetic theory, in the limit $\frac{v_{Te}^2 k_\perp^2}{\omega_{ce}^2} \ll 1$, is

$$\omega^2 = \omega_{UH}^2 + \frac{3v_{Te}^2 k_\perp^2 \omega_{pe}^2}{\omega^2 - 4\omega_{ce}^2}.$$

For $\omega \approx \omega_{UH}$, this becomes

$$\omega^2 \approx \omega_{UH}^2 + \frac{3v_{Te}^2 k_\perp^2 \omega_{pe}^2}{\omega^2 - 3\omega_{ce}^2}.$$

Upper hybrid waves propagating perpendicularly to the magnetic field are not Landau damped. They are one of many undamped **electron Bernstein modes**, described by kinetic theory. The dispersion relation for these modes is given by[1]:

$$1 + \frac{\omega_{pe}^2 e^{-\lambda}}{\omega_{ce}^2 \sin(\pi\Omega)} \int_0^\pi \sin(\psi\Omega)\sin(\psi) e^{-\lambda\cos(\psi)}\, d\psi = 0,$$

where $\lambda = \frac{k_\perp^2 v_{Te}^2}{\omega_{ce}^2}$ and $\Omega = \frac{\omega}{\omega_{ce}}$. It is convenient to write Equation (6.5) as

$$1 + \chi_e = 0,$$

where $\chi_e \equiv -\frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2 - \gamma_e v_{Te}^2 k_\perp^2}$ is the **electron susceptibility**, which is the response of plasma electrons to the electrostatic potential. This comes from expressing the electron density in terms of the potential in Poisson's equation. If the ion motion is taken into account, then by inserting the expression for the ion density in terms of the potential into Poisson's equation, one gets

$$1 + \chi_e + \chi_i = 0,$$

where $\chi_i = -\frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2 - \gamma_i v_{Ti}^2 k_\perp^2}$. For a plasma with multiple ion species, one has $1 + \chi_e + \chi_{i1} + \chi_{i2} + \ldots = 0$. For plasma waves with intermediate frequencies;

[1] F. W. Crawford and J. A. Tataronis. J. Appl. Phys. **36**:2930, 1965

$\omega_{ci}^2 \ll \omega^2 \ll \omega_{ce}^2$, and for $\omega_{ce}^2 \gg \gamma_e v_{Te}^2 k_\perp^2$ and $\omega^2 \gg \gamma_i v_{Ti}^2 k_\perp^2$, the electron and ion susceptibilities become

$$\begin{aligned} \chi_e &= \frac{\omega_{pe}^2}{\omega_{ce}^2} \\ \chi_i &= -\frac{\omega_{pi}^2}{\omega^2}, \end{aligned} \tag{6.6}$$

where the magnetic field has been neglected. These give the dispersion relation for **lower hybrid waves** as

$$1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2} = 0.$$

Solving for ω, one gets

$$\omega = \frac{\omega_{pi}\omega_{ce}}{\omega_{UH}}.$$

For $\omega_{pe}^2 \gg \omega_{ce}^2$ the oscillations occur at the **lower hybrid frequency**:

$$\begin{aligned} \omega &= \frac{\omega_{pi}\omega_{ce}}{\omega_{pe}} \\ &= \sqrt{\frac{m_e}{m_i}}\omega_{ce} \\ &= \sqrt{\omega_{ci}\omega_{ce}} \\ &\equiv \omega_{LH}. \end{aligned}$$

The lower hybrid waves exist for waves almost perfectly perpendicular to the magnetic field. In this case, the electrons cannot stream to neutralise the plasma, but instead they perform $\mathbf{E}\times\mathbf{B}$- and polarisation drifts perpendicular to the magnetic field, while the ions oscillate in the electrostatic field.

If the wave is only slightly oblique, i.e. $k_\parallel \ll k_\perp$ but still $\frac{k_\parallel}{k_\perp} > \sqrt{\frac{m_e}{m_i}}$ where $k_\parallel = k_\perp$, the electrons can stream along the magnetic field and neutralise the plasma. In this case there also exists a few low-frequency waves near the ion cyclotron frequency. The electrons can be treated as inertialess and therefore Equation (6.6) can be replaced by

$$\chi_e = \frac{\omega_{pe}^2}{v_{Te}^2 k^2} = \frac{1}{\lambda_{De}^2 k^2}.$$

Using the susceptibility for cold ions,

$$\chi_i = -\frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2},$$

one gets the dispersion relation for electrostatic ion cyclotron waves to be

$$\omega^2 = \omega_{ci}^2 + \frac{c_s^2 k^2}{1 + \lambda_{De}^2 k^2},$$

where $c_s = \sqrt{\frac{k_B T_e}{m_i}}$ is the cyclotron speed.

For reference, the ion and electron fluid susceptibilities for arbitrary propagation of electrostatic waves in a magnetic field are given by

$$\chi_{i,e} = -\frac{\omega_{pi,e}^2 (k^2 \omega^2 - k_\parallel^2 \omega_{ci,e}^2)}{k^2 [\omega^2 (\omega^2 - \omega_{ci,e}^2) - \gamma_{i,e} v_{Ti,e}^2 (k^2 \omega^2 - k_\parallel^2 \omega_{ci,e}^2)]},$$

where $k_\parallel = k_z$ in the geometry is the wavevector component parallel to the magnetic field, and $k = \sqrt{k_\perp^2 + k_\parallel^2}$ is the total length of the wavevector. From the equations for ion and electron susceptibilities, one can derive the various special cases of electrostatic waves with appropriate choices of $\gamma_{i,e}$ and being aware that kinetic effects can be important for warm plasmas.

6.4.4 Whistler/Helicon Waves

Consider a low-frequency electromagnetic wave which exists in a plasma in which the displacement current is neglected and the ions are assumed to be stationary. This can be due to the wave period being much shorter than the ion time-scale, which gives rise to **whistler waves**, or due to that the ions are prevented to move because because of collisions giving rise to low-frequency helicon waves. In this case, Ampère-Maxwell's law reads

$$\nabla \times \mathbf{B}_1 = -\mu_0 e n_0 \mathbf{v}_{e1} + \frac{1}{c^2} \frac{\partial \mathbf{E}_1}{\partial t}.$$

A second assumption is that the wave speed is much lower than the speed of light (this is true for $\omega_{pe} \gg \omega_{ce}$ but not necessarily for $\omega_{ce} > \omega_{pe}$). The displacement current can then be neglected, which gives Ampère's law, namely

$$\nabla \times \mathbf{B}_1 = -\mu_0 e n_0 \mathbf{v}_{e1}. \tag{6.7}$$

Electron density fluctuations have been explicitly ignored, which follows from the electron continuity equation by using the fact that taking the divergence of both sides of Ampère's law gives $\nabla \cdot n\mathbf{v}_{e1} = 0$. To obtain the electron velocity, one uses the electron momentum equation;

$$\frac{\partial \mathbf{v}_{e1}}{\partial t} = -\frac{e}{m_e}\mathbf{E}_1 - \mathbf{v}_{e1} \times \hat{\mathbf{z}}\omega_{ce} - \gamma_e v_{Te}^2 \frac{\nabla n_{e1}}{n_0}.$$

Next, assume that the electrons are inertialess due to the wave frequency being much smaller than the electron cyclotron frequency. Having already neglected electron density fluctuations;

$$0 = -\frac{e}{m_e}\mathbf{E}_1 - \mathbf{v}_{e1} \times \hat{\mathbf{z}}\omega_{ce}. \tag{6.8}$$

Finally, the electric and magnetic fields are related by Faraday's law, namely

$$\frac{\partial \mathbf{B}_1}{\partial t} = -\nabla \times \mathbf{E}_1. \tag{6.9}$$

Using Equation (6.8) to eliminate $\mathbf{E}_1$ in Equation (6.9) and using Equation (6.7) to eliminate $\mathbf{v}_{e1}$ gives the equation for the magnetic field as

$$\frac{\partial \mathbf{B}_1}{\partial t} = \frac{m_e \omega_{ce}}{\mu_0 e^2 n_0} \nabla \times \left(\hat{\mathbf{z}} \times (\nabla \times \mathbf{B}_1)\right).$$

As the cyclotron frequency is given by $\omega_{ce} = \frac{eB_0}{m_e}$, the masses cancel and hence the magnetic wave does not depend on the electron mass. Using the relation $\mu_0 = \frac{1}{\epsilon_0 c^2}$ and $\omega_{pe}^2 = \frac{e^2 n_0}{\epsilon_0 m_e}$, the equation for the magnetic wave can be written

$$\frac{\partial \mathbf{B}_1}{\partial t} = \frac{c^2 \omega_{ce}}{\omega_{pe}^2} \nabla \times \left(\hat{\mathbf{z}} \times (\nabla \times \mathbf{B}_1)\right)$$

or

$$\frac{\partial \mathbf{B}_1}{\partial t} = \lambda_e^2 \omega_{ce} \nabla \times \left(\hat{\mathbf{z}} \times (\nabla \times \mathbf{B}_1)\right), \tag{6.10}$$

where $\lambda_e = \frac{c}{\omega_{pe}}$ is the **electron skin depth** or **electron inertial length**. The dispersion relation (assuming $\frac{\partial}{\partial t} \to -i\omega$ and $\nabla \to i\mathbf{k}$) can be obtained by writing the components and eliminating the x-, y-, and z-components of $\mathbf{B}_1$:

$$\omega^2 = \lambda_e^4 k^2 k_z^2 \omega_{ce}^2.$$

Whistler waves are often created by thunderstorms in the equatorial region; they propagate along the magnetic field lines and can be observed in the northern hemisphere or by satellites as waves in the audible range (a few kilohertz). Since the waves with higher frequencies propagate faster, i.e. the group velocity is higher than the lower frequencies, the whistlers created by lightning give rise to a descending, whistle-like tone. For higher frequencies, one needs to take the electron inertia into account, hence replacing Equation (6.8) by

$$\frac{\partial \mathbf{v}_{e1}}{\partial t} = -\frac{e}{m_e}\mathbf{E}_1 - \mathbf{v}_{e1} \times \hat{\mathbf{z}}\omega_{ce}$$

then Equation (6.10) is modified to become

$$\frac{\partial}{\partial t}\left(\mathbf{B}_1 + \lambda_e^2 \nabla^2 \mathbf{B}_1\right) = \lambda_e^2 \omega_{ce} \nabla \times \left(\hat{\mathbf{z}} \times (\nabla \times \mathbf{B}_1)\right).$$

The dispersion for this case is then

$$\omega^2 = \frac{\lambda_e^4 k^2 k_z^2 \omega_{ce}^2}{\left(1 + \lambda_e k^2\right)^2}.$$

For small wavenumbers, i.e. $\lambda_e^2 k^2 \ll 1$, the original dispersion relation for inertialess electrons is retained, whereas for large wavenumbers, i.e. $\lambda_e^2 k^2 \gg 1$, there is an electron cyclotron resonance:

$$\omega^2 = \frac{k_z^2 \omega_{ce}^2}{k^2},$$

which is valid for $\omega_{ce}^2 \ll \omega_{pe}^2$.

6.5 High-Frequency Waves in Magnetised Plasmas

To study the general behaviour of high-frequency electromagnetic waves in magnetised plasmas consider the cold plasma equation of motion, namely

$$\frac{\partial \mathbf{v}_{e1}}{\partial t} = -\frac{e}{m_e}\mathbf{E}_1 - \mathbf{v}_{e1} \times \hat{\mathbf{z}}\omega_{ce},$$

which is coupled with the Faraday and Ampère-Maxwell laws;

$$
\begin{aligned}
\frac{\partial \mathbf{B}_1}{\partial t} &= -\nabla \times \mathbf{E}_1 \\
\nabla \times \mathbf{B}_1 &= -\mu_0 e n_0 \mathbf{v}_{e1} + \frac{1}{c^2}\frac{\partial \mathbf{E}_1}{\partial t}.
\end{aligned}
$$

This is a very simple model of a plasma, which is only valid when ions can be considered to be stationary, and when thermal and collisional effects are neglected. Still, the physics is surprisingly rich and complicated from this model. By assuming plane waves, the special cases of perpendicular and parallel propagation to the magnetic field, followed by the general case of oblique propagation will be discussed.

6.5.1 Perpendicular Propagation

Ordinary (O) mode waves propagate with the electric field parallel to the magnetic field lines. Due to this, the electrons are also accelerated parallel to the magnetic field and there is no influence of the magnetic field on the propagation of the O mode wave. The dispersion relation is the same as for an unmagnetised plasma;

$$\omega^2 = c^2 k_\perp^2 + \omega_{pe}^2.$$

Extraordinary (X) mode waves propagate with the electric field, perpendicular to the magnetic field. It has two branches; one high-frequency **fast X mode** which can propagate out of the plasma as light, and one lower-frequency **slow X mode** (often called the **Z mode**), which has an electrostatic resonance at the upper hybrid wave at large wavenumbers;

$$\left(\omega^2 - \omega_{pe}^2\right)\left(\omega^2 - c^2 k_\perp^2 - \omega_{pe}^2\right) - \omega_{ce}^2\left(\omega^2 - c^2 k_\perp^2\right) = 0. \tag{6.11}$$

At the cutoffs $k_\perp = 0$, the fast X mode is right-hand circularly polarised, while the Z mode is left-hand circularly polarised with respect to the magnetic field. For a given frequency, the X mode has the cutoff at the lowest electron density, followed by the O made and Z mode at higher densities.

6.5.2 Parallel Propagation

The **Langmuir wave** is an electrostatic wave that propagates with the electric field along the magnetic field line and is therefore not influenced by the magnetic field.

Left-hand (L) mode waves propagate with the electric field rotating in the left-hand direction perpendicular to the magnetic field lines.

Right-hand (R) mode waves propagate with the electric field rotating in the right-hand direction perpendicular to the magnetic field. It has two branches; one high frequency branch which frequencies higher than the L mode, which can propagate out of the plasma as light, and the low-frequency **whistler wave**, which exists only in the plasma. The dispersion relation reads

$$\omega^2 \left(\omega^2 - c^2 k_\parallel^2 - \omega_{pe}^2\right)^2 - \omega_{ce}^2 \left(\omega^2 - c^2 k_\parallel^2\right)^2 = 0.$$

At the respective cutoffs $k_\parallel = 0$, the L mode and R mode have the same cutoff frequency and polarisation as the Z mode and the fast X mode, respectively.

6.5.3 Oblique Propagation

The general case of oblique propagation is given by the Appleton-Hartree dispersion relation, which is often solved in terms of the wavenumber and is written as

$$\frac{k^2 c^2}{\omega^2} = 1 - \frac{2\omega_{pe}^2 \left(\omega^2 - \omega_{pe}^2\right) \omega^{-2}}{2\left(\omega^2 - \omega_{pe}^2\right) - \omega_{ce}^2 \sin^2(\theta) \pm \omega_{ce}\Delta},$$

where $\Delta = \sqrt{\omega_{ce}^2 \sin^4(\theta) + 4\omega^{-2}\left(\omega^2 - \omega_{pe}^2\right)^2 \cos^2(\theta)}$. The above can also be written

$$\begin{aligned}\omega^2 &\left(\omega^2 - \omega_{pe}^2\right)\left(\omega^2 - c^2k^2 - \omega_{pe}^2\right)^2 \\ &- \omega_{ce}^2 \left(\omega^2 - c^2k^2\right)\left(\omega^2\left(\omega^2 - c^2k^2 - \omega_{pe}^2\right) + c^2k^2\omega_{pe}^2\cos^2(\theta)\right) = 0.\end{aligned}$$

For oblique angles the R mode connects smoothly to the X mode at perpendicular propagation, whereas the L mode splits into the Z mode and O mode branches. The L-O mode has a cutoff $\omega = \omega_{pe}$ at $k = 0$, while the Z mode has a cutoff given by Equation (6.11) at $k_\perp = 0$. The Z mode converges to the electrostatic upper hybrid/Langmuir mode in the limit $\lambda_e k \gg 1$ and the oblique whistler wave connects to a low frequency electrostatic mode in the limit $\lambda_e k \gg 1$.

6.5.4 Typical Electron Time- and Length-Scales

It has been seen that high-frequency electromagnetic waves in magnetised plasmas can be characterised by the plasma frequency, ω_{pe}, and the electron cyclotron frequency, ω_{ce}, the electron skin depth, $\lambda_e = \frac{c}{\omega_{pe}}$, the X and O modes converging to the light vacuum speed c for large wavenumbers and the whistler waves having a lower typical speed, $\frac{c\omega_{ce}}{\omega_{pe}}$, when $\omega_{ce}^2 \ll \omega_{pe}^2$. Some typical values of $\frac{\omega_{ce}}{\omega_{pe}}$ include

Environment	Value
Space plasmas in the Earth's ionosphere (above 300 km)	$\frac{1}{10} \to \frac{1}{5}$
Solar winds	$\frac{1}{30}$
TOKAMAKs	1

6.6 Low-Frequency Electromagnetic Waves

Low-frequency magnetohydrodynamic (MHD) waves at around 1 Hz or below are sometimes observed in connection with solar storms etc. Shear Alfvén waves[2] can carry large amounts of energy and lead to disruptions of communications and the electrical grid on Earth. The simplest model for such waves are based on the inertialess, cold electron momentum equation

$$0 = \mathbf{E}_1 + \mathbf{v}_{e1} \times \mathbf{B}_0;$$

the cold ion momentum equation

$$\frac{\partial \mathbf{v}_{i1}}{\partial t} = \frac{e}{m_i} \left(\mathbf{E}_1 + \mathbf{v}_{i1} \times \mathbf{B}_0 \right);$$

[2] H. Alfvén, Nature **150**, pp. 405-406, 1942.

Faraday's law

$$\frac{\partial \mathbf{B}_1}{\partial t} = -\nabla \times \mathbf{E}_1;$$

and Ampère's law

$$\nabla \times \mathbf{B}_1 = \mu_0 e n_0 \left(\mathbf{v}_{i1} - \mathbf{v}_{e1}\right),$$

where the ion current is included but the displacement current has been neglected. Using the cold electron momentum equation to eliminate $\mathbf{E}_1$ in the cold ion equation gives

$$\frac{\partial \mathbf{v}_{i1}}{\partial t} = \frac{e}{m_i}\left(\mathbf{v}_{i1} - \mathbf{v}_{e1}\right) \times \mathbf{B}_0,$$

where the $(\mathbf{v}_{i1} - \mathbf{v}_{e1})$ term can be eliminated by using Ampère's law;

$$\begin{aligned}\frac{\partial \mathbf{v}_{i1}}{\partial t} &= \frac{1}{\mu_0 n_0 m_i}\left(\nabla \times \mathbf{B}_1\right) \times \mathbf{B}_0 \\ &= \frac{\left(\mathbf{B}_0 \cdot \nabla\right)\mathbf{B}_1}{\mu_0 n_0 m_i} - \frac{\nabla\left(\mathbf{B}_0 \cdot \mathbf{B}_1\right)}{\mu_0 n_0 m_i},\end{aligned} \tag{6.12}$$

where the first term on the right-hand side is due to **magnetic tension** and the second term is due to **magnetic pressure**. Alternatively, using the cold electron momentum equation to eliminate $\mathbf{E}_1$ in Faraday's law gives

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times \left(\mathbf{v}_{e1} \times \mathbf{B}_0\right),$$

where $\mathbf{v}_{e1}$ can be eliminated using Ampère's law:

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times \left(\left(\mathbf{v}_{i1} - \frac{\nabla \times \mathbf{B}_1}{\mu_0 e n_0}\right) \times \mathbf{B}_0\right). \tag{6.13}$$

This is known as the **induction equation**. The $\frac{\nabla\times\mathbf{B}_1}{\mu_0 e n_0}$ term is sometimes called the **Hall term** and consequently the model combining Equations (6.12) and (6.13) is a linear **Hall-MHD model**. Neglecting the Hall term gives the **ideal MHD**, which is the very simplest model for MHD waves. Taking the time derivative of Equation (6.13) and using Equation (6.12) to eliminate the time derivative of $\mathbf{v}_{i1}$ gives

$$\frac{\partial^2 \mathbf{B}_1}{\partial t^2} = \nabla \times \left(\left(\frac{\left(\mathbf{B}_0 \cdot \nabla\right)\mathbf{B}_1}{\mu_0 n_0 m_i} - \frac{\nabla\left(\mathbf{B}_0 \cdot \mathbf{B}_1\right)}{\mu_0 n_0 m_i} - \frac{1}{\mu_0 e n_0}\nabla \times \frac{\partial \mathbf{B}_1}{\partial t}\right) \times \mathbf{B}_0\right).$$

For waves with frequencies much lower than the ion cyclotron frequency, ω_{ci}, and wavelengths larger than the ion inertial length, $\lambda_i = \frac{c}{\omega_{pi}}$, one can usually neglect the Hall term. In this case one receives the ideal MHD model:

$$\frac{\partial^2 \mathbf{B}_1}{\partial t^2} = \nabla \times \left(\left(\frac{(\mathbf{B}_0 \cdot \nabla) \mathbf{B}_1}{\mu_0 n_0 m_i} - \frac{\nabla (\mathbf{B}_0 \cdot \mathbf{B}_1)}{\mu_0 n_0 m_i} \right) \times \mathbf{B}_0 \right).$$

Using the triple vector product identity $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{A} - \mathbf{B} (\nabla \cdot \mathbf{A})$:

$$\frac{\partial^2 \mathbf{B}_1}{\partial t^2} = \frac{(\mathbf{B}_0 \cdot \nabla)^2 \mathbf{B}_1}{\mu_0 n_0 m_i} + \frac{\mathbf{B}_0 \nabla^2 (\mathbf{B}_0 \cdot \mathbf{B}_1)}{\mu_0 n_0 m_i} - \frac{(\mathbf{B}_0 \cdot \nabla) \nabla (\mathbf{B}_0 \cdot \mathbf{B}_1)}{\mu_0 n_0 m_i}. \qquad (6.14)$$

This supports two wave modes; the magnetosonic and shear Alfvén waves.

6.6.1 Magnetosonic Waves

Assuming $\mathbf{B}_0$ and $\mathbf{B}_1$ are perpendicular to one another, taking the component parallel to $\mathbf{B}_0$ (taking the scalar product of Equation (6.14) with $\mathbf{B}_0$) obtains the wave equation for **magnetosonic waves**:

$$\frac{\partial^2 (\mathbf{B}_0 \cdot \mathbf{B}_1)}{\partial t^2} = v_A^2 \nabla^2 (\mathbf{B}_0 \cdot \mathbf{B}_1),$$

where $v_A = \sqrt{\frac{B_0^2}{\mu_0 n_0 m_i}}$ is the **Alfvén speed**, and $B_0^2 = \mathbf{B}_0 \cdot \mathbf{B}_0$. The magnetosonic waves are compressional waves similar to acoustic waves and propagate in all directions (except exactly parallel) to the magnetic field lines. The dispersion relation is then given by

$$\omega^2 = v_A^2 k^2.$$

Since the wave speed decreases with increasing plasma density, n_0, density maxima can work as wave guides for magnetosonic waves.

6.6.2 Shear Alfvén Waves

The wave equation for **shear Alfvén waves** is given by the component parallel to $\mathbf{B}_1$:

$$\frac{\partial^2 \mathbf{B}_1}{\partial t^2} = v_A^2 \frac{(\mathbf{B}_0 \cdot \nabla)^2 \mathbf{B}_1}{B_0^2}.$$

The dispersion relation is then

$$\omega^2 = v_A^2 k_\parallel^2.$$

The shear Alfvén waves propagate strictly along the magnetic field lines, and tend to follow curved magnetic field lines.

6.6.3 Electromagnetic Ion Cyclotron Waves and Whistlers

When the wave frequency approaches the ion cyclotron frequency, the Hall term must be kept. The dispersion relation for this case then reads

$$\left(\omega^2 - v_A^2 k^2\right)\left(\omega^2 - v_A^2 k_\parallel^2\right) - \frac{\omega^2}{\omega_{ci}^2} v_A^4 k^2 k_\parallel^2 = 0.$$

In this case, the shear Alfvén speed wave splits into two waves with different polarisation; whistler waves which has right-hand polarisation and electromagnetic ion cyclotron waves with left-hand polarisation. The ion cyclotron wave has frequencies below the ion cyclotron frequency and a resonance at the ion cyclotron frequency (typically around 40 Hz in the ionosphere). The whistler connects to the electron whistler with frequencies above the ion cyclotron frequency, and with a resonance when $\lambda_e k \gg 1$. The magnetosonic wave connects to the lower hybrid wave resonance and turns electrostatic when $\lambda_e k > 1$.

Chapter 7

Kinetic Theory

Plasmas can be described in several ways:

- **Single particle**: Essentially the behaviour of a charged particle in imposed fields, namely

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right).$$

- **Cold fluid**: Generalising single particle effects to construct a self-consistent current as the source of electromagnetic fields whilst avoiding thermodynamics;

$$\mathbf{J} = \sum_i n_i q_i \mathbf{v}_i.$$

- **Warm fluid**: Thermodynamic pressure is added to the cold fluid model, resulting in an imposed state where electromagnetic effects are compromised, e.g.

$$\mathbf{J} = \frac{\nabla \times \mathbf{B}}{\mu_0}.$$

- **Kinetic theory**: A comprehensive mathematical framework designed to model plasma particle dynamics, taking into account co-operative effects - this requires a statistical approach.

7.1 The Distribution Function

The distribution function is the basis of kinetic theory. Consider the average height, $\overline{h}$, of a discrete sample of N individuals with n_i of them being of height h_i, then

$$\overline{h} = \frac{1}{N}\sum_{i=1}^{N} n_i h_i.$$

The distribution function of a continuous sample of heights, $f(h)$, then gives the average height as

$$\overline{h} = \frac{\int h f(h)\ dh}{\int f(h)\ dh}.$$

A plasma has a very large number of particles, so that the description given by a distribution function is ideal, though multidimensional. At each point in time and space there is a complete distribution of velocities, given by

$$f = f(\mathbf{r}, \mathbf{v}, t).$$

The idea of 'average height' can be generalised when considering the plasma distribution function to construct **moments**, which are averages over the plasma:

Moment order	Average	Moment Equation
Zeroth	Plasma number density	$n(\mathbf{r}, t) = \int f(\mathbf{r}, \mathbf{v}, t)\ d\mathbf{u}$
First	Plasma fluid velocity	$\overline{\mathbf{v}}(\mathbf{r}, t) = \frac{1}{n}\int \mathbf{v} f(\mathbf{r}, \mathbf{v}, t)\ d\mathbf{v}$
Second	Plasma pressure tensor	$\mathbf{P}(\mathbf{r}, t) = m\int (\mathbf{v} - \overline{\mathbf{v}})(\mathbf{v} - \overline{\mathbf{v}})\, f(\mathbf{r}, \mathbf{v}, t)\ d\mathbf{v}$

The kinetic approach underpins all fluid (macroscopic) model construction, as well as being a statistically self-consistent description at the mesoscopic level, with some microscopic significance.

7.2 The Boltzmann Equation and Maxwellian Equilibrium

The fundamental (Boltzmann) equation describes how the distribution of charged particles is changed by mutual interactions in phase $(\mathbf{r}, \mathbf{v})$ space:

$$\begin{aligned}\left(\frac{\partial f}{\partial t}\right)_c &= \frac{\partial f}{\partial t} + \frac{\partial \mathbf{r}}{\partial t} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{\partial \mathbf{v}}{\partial t} \cdot \frac{\partial f}{\partial \mathbf{v}} \\ &= \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}}.\end{aligned}$$

The change in density is controlled by flux in and out of a closed surface which is given by the fluid continuity equation:

$$\partial_t \rho + \mathbf{v} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{v}.$$

However, the change in distribution function f is controlled by flux in and out of a closed hypersurface, with collision-scattering changing $(\mathbf{r}, \mathbf{v})$ configurations.

The key to the evolution is how the collisions are modelled:

- In **fully ionised plasmas** long-range Coulomb forces dominate; binary collisions (interactions between only two particles) are rare as the plasma evolves via many simultaneous weak interactions. There are two modelling approaches for this:
  - Vlasov
  - Fokker-Planck.
- **Partially ionised plasmas** include binary collisions, which may dominate. This must be able to describe both elastic and inelastic binary and long-range interactions. There are two models to cover this approach:
  - Boltzmann
  - Krook.

7.2.1 Equilibrium Solution

When there are no external forces acting upon a plasma, it relaxes to Maxwell-Boltzmann distribution, i.e. the Boltzmann collision (exponent) term is consistent with a plasma relaxing to a Maxwellian distribution in the absence of external forces;

$$f_0(v) = \left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} \exp\left(-\frac{mv^2}{2k_B T}\right).$$

Performing the moment calculations gives

Moment	Moment Equation
Plasma number density	$n = \int_{-\infty}^{\infty} f_0(v)\ d\mathbf{v}$
Mean (bulk) speed	$\overline{v} = \frac{1}{n}\int_{-\infty}^{\infty} v f_0(v)\ d\mathbf{v} = \left(\frac{8k_BT}{\pi m}\right)^{\frac{1}{2}}$
Mean energy	$\overline{\mathcal{E}} = \frac{1}{2}\frac{m}{n}\int_{-\infty}^{\infty} v^2 f_0(v)\ d\mathbf{v} = \frac{3}{2}k_BT$

There are also other useful forms of the Maxwellian:

Form	Equation
Energy probability function (EPF)	$g(\mathcal{E}) = \frac{2n}{\sqrt{\pi}(k_BT)^{\frac{3}{2}}}\mathcal{E}^{\frac{1}{2}} \exp\left(-\frac{\mathcal{E}}{k_BT}\right)$
Energy distribution function (EDF)	$g_p(\mathcal{E}) = \mathcal{E}^{-\frac{1}{2}} g(\mathcal{E})$

7.3 Vlasov

The Vlasov method sets all collisional kinematics to zero, i.e.

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m}\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial f}{\partial \mathbf{v}} = 0.$$

There are incorporated long-range ensemble interactions by self-consistent electromagnetic fields:

$$\nabla\cdot\mathbf{E}(\mathbf{r},t) = \sum_i \frac{q}{\epsilon_0}\int f_i(\mathbf{r},\mathbf{v},t)\ d\mathbf{v}$$

$$\nabla\times\mathbf{B}(\mathbf{r},t) = \mu_0\sum_i q\int \mathbf{v} f_i(\mathbf{r},\mathbf{v},t)\ d\mathbf{v} + \epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t},$$

where the first equation is the source of electrostatic field (charge separation) and the second equation is the source of the magnetic field (current flow).

The key point to the Vlasov approach is that the distribution function acts as an electromagnetic source, resulting in non-linear feedback and so the whole plasma behaves according to common fields. This method is widespread in the modelling of fully ionised plasmas as it yields important simple insights and agrees with the cold plasma model in an appropriate limit, whereby the plasma behaves like a dielectric, resulting in Landau damping (5.1);

$$\mathcal{E}(k,\omega) = 1 + \frac{e^2}{\epsilon_0 mk} \int \frac{\frac{\partial f_0}{\partial v_z}}{\omega - kv_z} \, d\mathbf{v}.$$

Linearising the Vlasov model's equations gives

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{r}} + \frac{e}{m}\mathbf{E} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0;$$
$$\nabla \cdot \mathbf{E} = \frac{e}{\epsilon_0} \int f_1 \, d\mathbf{v};$$
$$\mathcal{E}_r(k,\omega) = 1 + \frac{e^2}{\epsilon_0 mk} \int \frac{\frac{\partial f_0}{\partial v_z}}{\omega - kv_z} \, d\mathbf{v}.$$

There is a singularity at $\omega = kv_z$, hence this can be expanded around the pole as

$$(\omega - kv_z)^{-1} = \frac{1 + \frac{kv_z}{\omega} + \frac{k^2 v_z^2}{\omega^2} + \dots}{\omega}.$$

This then allows integration by parts on the second term so that

$$\mathcal{E}_r(k,\omega) = 1 - \frac{\omega_p^2}{\omega^2},$$

which is the cold plasma dielectric coefficient of cold plasma oscillation. Furthermore, integration by parts on the fourth term gives

$$\omega^2 = \omega_p^2 + 3k^2 \frac{k_B T}{m},$$

which describes the warm plasma electrostatic waves.

In order to bypass the singularity, one must implement contour integration in the complex plane, which potentially results in a complex frequency:

$$\begin{aligned}\mathcal{E}(k,\omega) &= 1 + \frac{e^2}{\epsilon_0 mk}\int \frac{\frac{\partial f_0}{\partial v_z}}{\omega - kv_z}\,d\mathbf{v}\\ &= 1 + \frac{\omega_p^2}{nk^2}\int \frac{f'}{v - v_p}\,dv.\end{aligned}$$

The singularity then exists at $v = \frac{\omega}{k}$, resulting in the integral becoming

$$\int_c \frac{f'}{v - v_p}\,dv = P\int \frac{f'}{v - v_p}\,dv + 2\pi i f'(v_p),$$

where the first term is the principle part and the second term is the "residue", which is a damping term. The real, r, and imaginary, i, dispersion relations for Landau damped Langmuir waves are given by

$$\begin{aligned}\omega &= \omega_r + i\omega_i;\\ \omega_r &= \omega_{pe}\sqrt{1 + 3k^2\lambda_D^2} \approx \omega_{pe}\left(1 + \frac{3}{2}k^2\lambda_D^2\right),\\ \omega_i &= -\left(\frac{\pi}{8}\right)^{\frac{1}{2}}\frac{\omega_{pe}}{k^3\lambda_D^3}\exp\left(\frac{3}{2} - \frac{1}{2k^2\lambda_D^2}\right).\end{aligned}$$

The real component of this is the same as the Bohm-Gross relation earlier derived using the two-fluid model (Equation (6.2)), however an imaginary component - the Landau damping - has now surfaced. The two-fluid model could not calculate this result as it is a result of the internal dynamics of the electron velocity space, to which the model was oblivious. The physical interpretation of this relation is that electrons that are moving close to the phase velocity of the wave will gain energy from it, as will electrons moving slower than the phase speed, however electrons will lose energy if they are travelling faster than the phase speed.

7.4 Cross Sections and Types of Reaction

The collision cross-section, σ_c, is the interaction area for a collision, e.g. for hard spheres this would be

$$\sigma_c = \pi a^2.$$

The mean free path, λ_{mfp}, is the average distance between collisions, i.e. for a material of number density n

$$\lambda_{\text{mfp}} = \frac{1}{n\sigma_c}.$$

The collisional frequency, ν_c, is the average rate for collisions to occur;

$$\nu_c = n\sigma_c v.$$

In general, collision cross-sections and mean free paths are dependent on velocity, as well as target particles having normal distributions of relative speeds.

7.4.1 Binary Coulomb Interaction

The scatter of charged particles is due to the Coulomb potential. Consider two particles, each of charge q_i, mass m_i and relative speed v_0 at scattering angle θ_c, the Rutherford cross section, σ_c, is given by

$$\sigma_c(v_0, \theta_c) = \left(\frac{q_1 q_2}{8\pi\epsilon_0 m_r v_0^2 \sin^2\left(\frac{1}{2}\theta_c\right)} \right)^2.$$

Incorporating the Boltzmann collision term gives

$$\left(\frac{\partial f_1}{\partial t}\right)_c = \int \left(\underbrace{f_1' f_2'}_{\substack{\text{post-}\\\text{collision}}} - \underbrace{f_1 f_2}_{\substack{\text{pre-}\\\text{collision}}} \right) \|v_1 - v_2\| \, \sigma_c \; dv_2 \; d\Omega.$$

The plasma rapidly relaxes into the Maxwellian via strong binary interactions, however the collisions are binary only - as well as being uncorrelated and restricted to coarse-grained variations in the distribution function - which are rather restrictive assumptions. Instead, plasma evolution is usually the result of many small interactions.

7.5 Fokker-Planck

Taking into account many weak, simultaneous interactions, the Fokker-Planck collision term is based upon Brownian motion:

$$\left(\frac{\partial f}{\partial t}\right)_c = \underbrace{-\frac{\partial\left(f\langle\Delta\mathbf{v}\rangle\right)}{\partial\mathbf{v}}}_{\text{friction}} + \underbrace{\frac{1}{2}\frac{\partial^2\left(f\langle\Delta\mathbf{v}\Delta\mathbf{v}\rangle\right)}{\partial\mathbf{v}\partial\mathbf{v}}}_{\text{diffusion}},$$

where

$$\langle\Delta\mathbf{v}\rangle = \frac{1}{\Delta t}\int \Psi\Delta\mathbf{v}\ d(\Delta\mathbf{v});$$
$$\langle\Delta\mathbf{v}\Delta\mathbf{v}\rangle = \frac{1}{\Delta t}\int \Psi\Delta\mathbf{v}\Delta\mathbf{v}\ d(\Delta\mathbf{v}),$$

where Ψ is the probability function; the likelihood that a particle initially with velocity $\mathbf{v}$ acquires an increment $\Delta\mathbf{v}$ in time Δt via a small-angle scattering process. The velocity of a group of particles may be changed by many weak interactions, known as **dynamical friction**, as well as being able to spread out in velocity space, known as **velocity diffusion**. The scattering process is independent of time or history, hence it is a **Markovian process**.

The Fokker-Planck collision term can be written in a simplified way:

$$\left(\frac{\partial f}{\partial t}\right)_c = \Gamma_p\left[-\frac{\partial\left(f\frac{\partial\mathcal{H}}{\partial\mathbf{v}}\right)}{\partial\mathbf{v}} + \frac{1}{2}\frac{\partial^2\left(f\frac{\partial^2\mathcal{G}}{\partial\mathbf{v}\partial\mathbf{v}}\right)}{\partial\mathbf{v}\partial\mathbf{v}}\right],$$

where

$$\Gamma_p = \frac{q^2q_{sc}^2}{4\pi\epsilon_0^2m^2}\ln(\Lambda),$$
$$\mathcal{G} = \int f_{sc}(\mathbf{v}_{sc})|\mathbf{v}-\mathbf{v}_{sc}|\ d\mathbf{v}_{sc},$$
$$\mathcal{H} = \frac{m+m_{sc}}{m}\int\frac{f_{sc}(\mathbf{v}_{sc})}{|\mathbf{v}-\mathbf{v}_{sc}|}\ d\mathbf{v}_{sc},$$

and $\ln(\Lambda)$ is the Coulomb logarithm.

7.6 Krook

The BGK Model (Krook approximation) is a very simple construction which assumes that the plasma is close to an identifiable equilibrium. This is a close approximation to the Boltzmann collision term:

$$\left(\frac{\partial f}{\partial t}\right)_c = \frac{f - f_0}{\tau_c},$$

where τ is the time of a single collision, which yields simple insights into transport coefficients, however it does not necessarily conserve particle number, momentum or energy. It is very good for electron-scattering models in a minority plasma. Assuming Maxwellian equilibrium, the constant collision frequency, ν_c, is given by

$$\nu_c = \frac{1}{\tau_c}.$$

Other quantities related to this frequency include:

Quantity	Equation
Plasma conductivity	$\sigma = \frac{ne^2}{m\nu_c}$
Particle mobility	$\mu = \frac{e}{m\nu_c}$
Thermal conductivity	$K = \frac{5nk_B^2 T}{2m\nu_c}.$

The Krook operator can be exploited in advanced fluid simulations by using a lattice Boltzmann method.

7.7 Reaction Rates in Chemical Kinetics

Atoms and molecules possess complex sets of electronic configurations. Energy can be delivered to the atom/molecule via photons or collisions, causing internal excitation. If enough energy is delivered then an atomic or molecular electron is liberated, creating an ion, the probability of which is given by the cross-section.

Consider an incident particle, e.g. an electron, travelling toward a target particle, e.g. an atom or molecule. Assuming an elastic collision, the target is not excited and instead it acquires momentum because the energy transferred did not reach the threshold energy of the atom/molecule. The reaction can be written

$$\mathrm{e} + \mathrm{A} \rightarrow \mathrm{e} + \mathrm{A}.$$

However, in an inelastic collision the electron slows and the atom/molecule is internally excited:

$$\mathrm{e}_{\text{fast}} + \mathrm{A} \rightarrow \mathrm{e}_{\text{slow}} + \mathrm{A}^*.$$

This excitation can be lost by losing an electron (ionisation) or releasing a photon (de-excitation). However, the target can remain excited provided the relaxation is forbidden (metastable).

Despite their names, the chemical kinetics of a reaction are carried out within the fluid limit, not the kinetic limit. The reaction rate, k, is the average rate of interaction, $\langle \sigma v \rangle$, hence is given by

$$k = \frac{\int f(\mathbf{v}, t)\sigma(\mathbf{v})v \; d\mathbf{v}}{\int f(\mathbf{v}, t) \; d\mathbf{v}}.$$

The rate coefficient can then be used in rate equations, e.g. the rate of change of number density of species 1, given in terms of the rate coefficient for the reaction between species 1 and species 2, is given by

$$\frac{\partial n_1}{\partial t} = k n_1 n_2.$$

The general modified (Arrhenius) form for a chemical process is then given by

$$k = A \left(\frac{T}{T_0} \right)^n \exp \left(-\frac{E_a}{k_B T} \right),$$

where E_a is the activation energy.

Chapter 8

Driven Plasma

Technological plasmas are often driven, resulting in relaxation timescales being on the order of external driving frequencies. Furthermore, plasmas are often minority species, meaning that collisions with neutrals are very important. This is as well as being bounded, resulting from large electric fields being present. It is therefore important to formulate a plasma model which incorporates these features.

8.1 Margenau Distribution

The Margenau distribution gives a general form of electron distribution function for gas-plasma mixture in the present of an oscillating electric field[1]:

$$f = Ae^{-W};$$
$$W = \int_0^v dv\ mv \left(\underbrace{k_B T_g}_{E_{th}} + \underbrace{\frac{2e^2 E_{\text{rms}}^2}{3m\left(\nu^2+\omega^2\right)\xi}}_{E_{\text{drive}}} \right)^{-1},$$

[1]Desloge, Matthysse, Margenau and McDaniel. *Collision Phenomena in Ionised Gases*, Phys. Rev. **112**:1437, 1958.

where ν is the collision frequency for elastic electron-neutron interactions, ω is the driving frequency, and ξ is the energy loss paramter, i.e. the average energy transfer per collision.

Four cases can be readily identified:

1. The thermal energy of the gas is dominant, $E_{th} \gg E_{\text{drive}}$, resulting in a Maxwellian distribution:

$$W = \frac{\frac{1}{2}mv^2}{k_B T_g}.$$

2. An harmonic electric field in which thermal motion is negligible, $E_{th} \ll E_{\text{drive}}$, resulting in a modified Druyvesteyn distribution, which has fewer high-energy particles than the equivalent Maxwellian with the same total energy content:

$$W = \frac{3}{2}\xi\left(\frac{\frac{1}{2}mv^2}{eE_{\text{rms}}\lambda_{\text{mfp}}}\right) + \frac{3}{4}\xi\left(\frac{m_e v\omega}{eE_{\text{rms}}}\right)^2.$$

3. A DC electric field ($\omega = 0$) is dominant, $E_{\text{rms}} = E_{DC} \gg \frac{\sqrt{k_B T}}{e}$, resulting in a Druyvesteyn distribution, which has fewer high energy electrons than a Maxwellian distribution; this is important for ionisation/excitation rates:

$$W = \frac{3}{2}\xi\left(\frac{\frac{1}{2}mv^2}{eE_{DC}\lambda_{\text{mfp}}}\right)^2.$$

The rate of change of the mean electron energy, U_e, is simple related to the mean gas target energy, U_g:

$$\frac{dU_e}{dt} = -\xi\nu_c\left(U_e - U_g\right).$$

Therefore if $U_e < U_g$ then the electrons are heated, and vice-versa.

4. A high-frequency electric field, resulting in another Maxwellian but with a corrected 'temperature' ($\omega \gg \nu$):

$$W \approx \frac{\frac{1}{2}mv^2}{k_B T_{eq}},$$

where

$$k_B T_{eq} = k_B T_g + \frac{2e^2 E_{\rm rms}^2}{3m\omega^2 \xi}.$$

The general expression for the distribution function is given by

$$F_0 = C \left(1 + \frac{\epsilon}{\beta_1 + \beta_2}\right)^{\beta_1} \exp\left(-\epsilon\right),$$

where

$$\epsilon = \frac{mu^2}{2k_B T_g},$$
$$\beta_1 = \frac{1}{3}\left(\frac{e^2 E_{rms}^2 \lambda_{\rm mfp}^2}{k_B^2 T_g^2 \xi}\right),$$
$$\beta_2 = \frac{m\lambda_{\rm mfp}^2 \omega^2}{2k_B T_g}.$$

When the electron temperature is significantly higher than that of the gas and the density is not too low, the equilibrium distribution function is Druyvesteyn.

8.2 Ionisation Equilibrium

In order to predict the degree of ionisation of a gas-plasma mix, one uses the thermal equilibrium as a starting point. For a system with discrete energy levels the degree of ionisation is given by

$$\frac{N_b}{N_a} = \frac{g_b}{g_a} \exp\left(-\frac{\Delta E}{k_B T}\right),$$

where $N_{a,b}$ are the numbers of particles in energy levels a and b which are of energies $E_{a,b}$, respectively, $\frac{g_b}{g_a}$ is the degeneracy of the energy level, the exponential is the Boltzmann factor, and $\Delta E = E_b - E_a$ is the difference in energy levels. This system is appropriate for an atom, but the theory must be extended for ionisation to cover atomic (ion-electron) ensembles.

Given a system in which an atom has excited states $E_1, E_2, \ldots$, its ion has ionisation stages $N_I, N_{II}, \ldots$ as well as excited states (Note: the Roman numeral subscripts indicate ionisation stages; N_I is neutral, N_{II} is first ionisation etc). In equilibrium the rate of atoms being ionised is the same as the rate of ions being de-excited, i.e.

$$A \Leftrightarrow A^+ + e^-.$$

The Saha equation (Equation (1.1)) gives the relative proportions of each ionisation stage as

$$\frac{n_{i+1}n_i}{n_e} = \underbrace{\frac{Z_{i+1}}{Z_i}}_{\substack{\text{partition}\\ \text{functions}}} \underbrace{g_e \left(\frac{2\pi m_e k_B T}{h^2}\right)^{\frac{3}{2}}}_{\text{electron term}} \exp\left(-\frac{\chi_i}{k_B T}\right),$$

where χ_i is the ionisation energy.

8.3 Waves in Kinetic Plasmas

Recalling the dielectric coefficient for an isotropic unmagnetised kinetic plasma;

$$\epsilon_r = 1 + \frac{e^2}{\epsilon_0 m k} \int \frac{\frac{\partial f_0}{\partial u_z}}{\omega - k u_z} \, d\mathbf{u},$$

this may be written in the form of a dispersion relation for electrostatic (dielectric coefficient vanishes) modes:

$$1 = \frac{\omega_p^2}{n k^2} \int \frac{f_0'}{u - v_p} \, du.$$

This results in not just plasma oscillation but Langmuir waves. If the equilibrium cannot be described in terms of a single Maxwellian distribution then the waves must be unstable. The damping term is proportional to the gradient of the equilibrium distribution in velocity space, hence a positive gradient in the distribution function can lead to a growth in the disturbance. A bump-in-tail instability is a special case of a two-stream instability.

In magnetised kinetic plasmas the magnetic field complicates the dielectric response, giving a dielectric tensor rather than a constant. Using a linearised Vlasov equation for small-amplitude waves gives

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \frac{\omega^2}{c^2} \epsilon \mathbf{E} = 0,$$

with dielectric tensor

$$\epsilon = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}.$$

8.3.1 Parallel to Magnetic Field

Parallel to the magnetic field the observable phenomena include:

1. **Langmuir waves**: electrostatic, longitudinal.

2. **Alfvén waves**: transverse, low-frequency electromagnetic dispersion:

$$\begin{aligned} \omega &= \omega_r + i\omega_i \\ \omega_r^2 &= \frac{k_\parallel c_a^2}{1 + \frac{c_a^2}{c^2}} \\ \omega_i &= -\frac{\omega_{pi}^2}{\left|k_\parallel\right| \overline{u}_i} \left(1 + \frac{c^2}{c_a^2}\right)^{-1} \exp\left(-\frac{m_i c_a^2}{2k_B T_i} \frac{\omega_{ci}^2}{\omega_T^2}\right). \end{aligned}$$

3. **Whistler waves**: electromagnetic, intermediate frequency dispersive modes: $\omega_{ci} \ll \omega \ll \omega_{ce}$:

$$\omega_i \approx -\frac{\omega_{pe}^2}{\left|k_\parallel\right| \left(\frac{2k_B T_i}{m_i}\right)^{\frac{1}{2}}} \left(1 + \frac{k_\parallel c^2}{\omega_r^2}\right)^{-1} \exp\left(-\frac{m_e \omega_{ce}^2}{2k_B T_e k_\parallel^2}\right).$$

4. **Cyclotron waves**: transverse, high-frequency near electron cyclotron frequency:

$$\omega \approx k_\parallel c - i\left(\frac{m_e c^2}{8\pi k_B T_e}\right)^{\frac{1}{2}} \frac{\omega_{pe}^2}{\omega_{ce}}.$$

8.3.2 Perpendicular to Magnetic Field

1. Ordinary mode: transverse electromagnetic, electric vector aligned along equilibrium magnetic field

$$\frac{k_\perp^2 c^2}{\omega^2} = 1 - \sum_s \frac{\omega_{ps}^2}{\omega} e^{-b_s} \sum_{n=-\infty}^{\infty} \frac{I_b(b_s)}{\omega - n\omega_{cs}},$$

where $b(s) = \frac{k_\perp k_B T}{m_s \omega_{cs}^2}$ and I_n are Bessel functions.

2. Extraordinary mode: (almost) transverse electromagnetic waves, with very complicated dispersion relation

$$\left[1 - \frac{k^2c^2}{\omega^2} - \sum_s \frac{\omega_{ps}^2}{\omega} \frac{e^{-b_s}}{b_s} \sum_{n=-\infty}^{\infty} \frac{n^2 I_n(b_s) + 2b_s^2 \left(I_n(b_s) - I_n'(b_s)\right)}{\omega - n\omega_{cs}}\right]$$
$$\times$$
$$\left[1 - \sum_s \frac{\omega_{ps}^2}{\omega} \frac{e^{-b_s}}{b_s} \sum_{n=-\infty}^{\infty} \frac{n^2 I_n(b_s)}{\omega - n\omega_{cs}}\right]$$
$$=$$
$$\left[\sum_s \frac{\omega_{ps}}{\omega} e^{-b_s} \sum_{n=-\infty}^{\infty} \frac{n \left(I_n(b_s) - I_n'(b_s)\right)}{\omega - n\omega_{cs}}\right]^2.$$

3. Bernstein mode: longitudinal electrostatic mode propagating perpendicular to magnetic field

$$k_\perp^2 = \sum_s \sum_{n=1}^{\infty} \frac{2n^2 \omega_{ps}^2 \omega_{cs}^2}{\omega^2 - n^2 \omega_{cs}^2} \frac{m_s}{k_B T_s} e^{-b_s} I_n(b_s).$$

8.4 Summary

The kinetic descriptions may be interactable for large-scale, complex plasmas; the moments of the Boltzmann equation are taken in order to get a fluid description. The general transfer equation is given by

$$\left[\frac{\partial}{\partial t}\left(n\langle\psi\rangle\right)\right]_c = \frac{\partial\left(n\langle\psi\rangle\right)}{\partial t} + \frac{\partial\left(n\langle\mathbf{u}\psi\rangle\right)}{\partial \mathbf{r}} \frac{nq}{m} \mathbf{E} \cdot \left\langle \frac{\partial\psi}{\partial \mathbf{u}} \right\rangle + \frac{nq}{m} \left\langle (\mathbf{u} \times \mathbf{B}) \cdot \frac{\partial \psi}{\partial \mathbf{u}} \right\rangle.$$

Model	Scale
Single particle	Microscopic
Kinetic description	Mesoscopic
Fluid models	Macroscopic

The first few moment equations of a purely ionised homogeneous plasma are:

$$
\begin{aligned}
0 &= \frac{\partial n_{i,e}}{\partial t} + \nabla \cdot (n_{i,e}\mathbf{v}) \\
-\mathbf{K} &= n_e m_e \left(\frac{\partial}{\partial t} + \mathbf{u}_e \cdot \nabla\right)\mathbf{u}_e + \nabla p_e + n_e e\left(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}\right) \\
\mathbf{K} &= n_i m_i \left(\frac{\partial}{\partial t} + \mathbf{u}_i \cdot \nabla\right)\mathbf{u}_i + \nabla p_i + n_i e\left(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}\right),
\end{aligned}
$$

where the first two equations are conservations of electrons and ions - which is not always true if there are sources or sinks present - the last two are equations for momenta, $p_{i,e}$ are scalar pressures and $\mathbf{K}$ are collisional terms highlighting momentum exchange which are given by

$$
\mathbf{K} - \int m_i \mathbf{u}\left(\frac{\partial f_i}{\partial t}\right)_c d\mathbf{u} = -\int m_e \mathbf{u}\left(\frac{\partial f_e}{\partial t}\right)_c d\mathbf{u}.
$$

The options for building a fluid model are:

- Neglect fluid velocity compared with bulk flow:
  Cold plasma model
- Retain full thermal physics, drop short timescales:
  Hydro plasma model
- Retain all timescales and thermodynamics:
  Who knows?