Doc Dave

MPhys Notes

Quantum Field Theory

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Contents

Chapter 1

Preliminaries

1.1 Why Study Quantum Field Theory?

Quantum field theory (QFT) combines earlier courses in classical fields, special relativity and quantum mechanics to unify particles and forces. Whereas previous quantum mechanics courses treat particles as waves, the basis of QFT is to treat particles as excitations of an underlying field, whose interactions can be visualised via Feynman diagrams.

1.2 Classical Systems

1.2.1 Classical Lagrangian Dynamics

The Lagrangian L of a system with generalised coordinates q_i is given by

$$L(q_i, \dot{q}_i) = T(\dot{q}_i) - V(q_i),$$

where T and V are respectively the kinetic and potential energies of the system and $\dot{q}_i \coloneqq \frac{\mathrm{d}q_i}{\mathrm{d}t}$. The equation of motion of a system given its Lagrangian is calculated using the **Euler-Lagrange equation**.

Definition 1.1: Euler-Lagrange Equation

The basis of the Euler-Lagrange equation in Lagrangian dynamics is to find the conditions for which a system's Lagrangian L extremises (maximises/minimises) the action $\mathcal{S}$ (its derivative is equal to zero), which are related by

$$\mathcal{S} \coloneqq \int_{t_1}^{t_2} L \, \mathrm{d}t.$$

In 1D, the Euler-Lagrange equation (derived later) is written

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$$

Example 1.1: Mass-spring system

The simplest macrosystem is a mass-spring system, whose Lagrangian is given by

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.$$

Applying the Euler-Lagrange equation to this Lagrangian then gives

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x}\right) + kx = 0 \\ &\Rightarrow m\ddot{x} = -kx, \end{aligned}$$

i.e. Hooke's law.

1.2.2 Hamiltonian Dynamics

The Hamiltonian of a system is defined by the Legendre transform

$$H(q_i, p_i) \coloneqq \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t),$$

where p_i are conjugate momenta given by $p_i \coloneqq \left.\frac{\partial L}{\partial \dot{q}_i}\right|_{\dot{q}_i=\dot{q}_i(q_i,p_i)}$. p_i and q_i are canonical coordinates in phase space used in the Hamiltonian formalism. These coordinates are defined such that they satisfy the following Poisson relationships:

$$\begin{aligned} \{q_i, q_j\} &= 0 \qquad \text{Cartesian positions} \\ \{p_i, p_j\} &= 0 \qquad \text{Cartesian momenta} \\ \{q_i, p_j\} &= \delta_{ij}, \end{aligned}$$

where the Poisson bracket is defined by

$$\{f_i, g_i\} \coloneqq \sum_{k=1}^{N} \left(\frac{\partial f_i}{\partial q_k} \frac{\partial g_j}{\partial p_k} - \frac{\partial f_i}{\partial p_k} \frac{\partial g_j}{\partial q_k} \right).$$

As a result, when two functions f and g have Poisson bracket $\{f, g\} = 0$, the functions have eigenvalues which can be measured simultaneously. The Poisson bracket in classical mechanics is analogous to the commutator in quantum mechanics, hence an important tool in moving from classical to quantum mechanics.

Hamilton's equations of motion are

$$\begin{aligned} \frac{\partial H}{\partial p_i} &= \dot{q}_i = \{q_i, H\} \\ -\frac{\partial H}{\partial q_i} &= \dot{p}_i = \{p_i, H\}. \end{aligned}$$

Proof 1.1: $\frac{\partial H}{\partial p_i} = \dot{q}_i; -\frac{\partial H}{\partial q_i} = \dot{p}_i$

As the Hamiltonian of a system is given by

$$H(q_i, p_i, t) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t),$$

a small variation in the Hamiltonian $\mathrm{d}H$ is then

$$\mathrm{d}H = \sum_i p_i \mathrm{d}\dot{q}_i + \mathrm{d}p_i \dot{q}_i - \frac{\partial L}{\partial q_i}\,\mathrm{d}q_i - \frac{\partial L}{\partial \dot{q}_i}\,\mathrm{d}\dot{q}_i - \frac{\partial L}{\partial t}\,\mathrm{d}t.$$

Using the definition of the conjugate momenta ($p_i = \frac{\partial L}{\partial \dot{q}_i}$), then from the Euler-Lagrange equation one gets $\frac{\partial L}{\partial q_i} = \dot{p}_i$. The equation for $\mathrm{d}H$ is hence

$$\mathrm{d}H = \sum_i p_i \mathrm{d}\dot{q}_i + \mathrm{d}p_i \dot{q}_i - \dot{p}_i \mathrm{d}q_i - p_i \mathrm{d}\dot{q}_i - \frac{\partial L}{\partial t}\,\mathrm{d}t,$$

which cancels to give

$$\mathrm{d}H = \sum_i \dot{q}_i \mathrm{d}p_i - \dot{p}_i \mathrm{d}q_i - \frac{\partial L}{\partial t}\,\mathrm{d}t.$$

Using this result with $\mathrm{d}H(q_i, p_i, t) = \sum_i \frac{\partial H}{\partial q_i}\mathrm{d}q_i + \frac{\partial H}{\partial p_i}\mathrm{d}p_i + \frac{\partial H}{\partial t}\mathrm{d}t$, then

$$\frac{\partial H}{\partial p_i} = \dot{q}_i; \qquad -\frac{\partial H}{\partial q_i} = \dot{p}_i; \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}\,.$$

The Hamiltonian of a system is its total energy, i.e. the sum of kinetic and potential energies;

$$H = T + V.$$

The Hamiltonian for a mass-spring system is then

$$\begin{aligned} H &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \\ &= \frac{p^2}{2m} + \frac{1}{2}kx^2. \end{aligned}$$

1.2.3 Functional Differentiation

As was met in previous courses, functional differentiation concerns rates of changes of functionals with respect to the function on which it depends.

Consider a set of discrete variables $\{x_i\}$. The derivative of one variable with respect to another variable is given by the Kronecker delta, i.e.

$$\frac{\mathrm{d}x_i}{\mathrm{d}x_j} = \delta_{ij} \Rightarrow \sum_i \left(\frac{\mathrm{d}}{\mathrm{d}x_j}\,x_i\right) = 1.$$

This method of discrete differentation can be extended to coordinates that vary continuously with time, i.e. $\frac{\delta q_i(t)}{\delta q_j(t')}$. As [causality?], the derivative of a particle with respect to itself at another time cannot be non-zero when the times are not equal, hence

$$\begin{aligned} \frac{\delta x_i(t)}{\delta x_i(t')} &= 0 \qquad t \neq t' \\ \frac{\delta x_i(t)}{\delta x_i(t')} &\neq 0 \qquad t = t'. \end{aligned}$$

The only function with this property is a Dirac delta, i.e. $\frac{\delta x_i(t)}{\delta x_i(t')} = \delta(t - t')$. As a result, the continuous derivative of two generalised coordinates is given by

$$\frac{\delta q_i(t)}{\delta q_j(t')} = \delta_{ij}\delta(t - t').$$

If i and j are taken to be continuous (or very large), i.e. there are many independent particles, the summation instead becomes an integral, i.e.

$$\int \mathrm{d}t' \ \frac{\delta x(t)}{\delta x(t')} = 1.$$

1.2.4 Action in Lagrangian Mechanics

The action $\mathcal{S}$ is a functional defined by

$$\mathcal{S}\left[q_i(t)\right] \coloneqq \int_{t_1}^{t_2} \mathrm{d}t \ L\left[q_i(t), \dot{q}_i(t)\right].$$

The action is the mapping of generalised coordinates from function space (e.g. $e^{i\omega t}, \sin(\omega t)$) to real space, i.e. $\mathcal{F} \ni q_i \mapsto \mathcal{S}[q_i(t)] \in \mathbb{R}$.

To analyse a small variation within a function f of a variable x, one expands as

$$f(x + \delta x) = f(x) + \delta x \cdot f'(x) + \mathcal{O}(\delta x^2).$$

This can be extended to a function of a vector as

$$f(\mathbf{x} + \delta\mathbf{x}) = f(\mathbf{x}) + \delta\mathbf{x} \cdot \boldsymbol{\nabla} f(\mathbf{x}) + \mathcal{O}(\delta x^2).$$

Hamilton's principle states that the actual motion of a system is determined by stationary points of the action, i.e. $\frac{\delta \mathcal{S}}{\delta q_i(t)} \stackrel{!}{=} 0$. This is an extension of Fermat's principle, which states that light travels along the route that minimises the travel time between the initial and end locations.

Consider a particle leaving a point q_1 at a time t_1 and arriving at another point q_2 at a later time t_2. There are an infinite number of paths along which the particle could travel, however it will only take the one that minimises the action. A path taken by the particle is a collection of locations q_i at each time t such that $t_1 \leq t \leq t_2$, i.e. a path can be denoted $q_i(t)$. A slightly different path is the same coordinates displaced by a small variation $\delta q_i(t)$, i.e. this slightly different path can be denoted $q_i + \delta q_i$, as shown in Figure 1.3. There is no variation at the end points as they are fixed, i.e. $\delta q_i(t_1) = \delta q_i(t_2) = 0$.

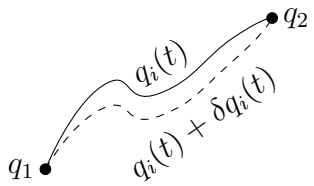


Figure 1.1: Two slightly different paths that could be taken by a particle.

The action for a path $q_i(t)$ is then given by

$$\mathcal{S}[q_i(t)] = \int_{t_1}^{t_2} \mathrm{d}t \; L(q_i(t), \dot{q}_i(t)).$$

Small variations in the Lagrangian result in small variations in the action $\delta\mathcal{S}$ which are usually non-zero. Hamilton's principle of least action ($\delta\mathcal{S} = 0$) can be used to derive the equation of motion of the path taken.

Proof 1.2: Deriving the equation of motion from Hamilton's principle of least action I

Expanding the action for a path with a slight variation gives

$$\mathcal{S}[q_i(t) + \delta q_i(t)] = \mathcal{S}[q_i(t)] + \underbrace{\sum_j \int_{t_1}^{t_2} \mathrm{d}t' \; \frac{\delta \mathcal{S}[q_i(t)]}{\delta q_j(t')} \delta q_j(t')}_{=\delta\mathcal{S}} + \mathcal{O}(\delta q_i^2).$$

From the definition of the action, i.e. $\mathcal{S} = \int_{t_1}^{t_2} \mathrm{d}t\ L$, one derives a small variation in action to be

$$\delta\mathcal{S} = \int_{t_1}^{t_2} \mathrm{d}t\ \left(\delta q_i \frac{\partial L}{\partial q_i} + \delta\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right).$$

Using the product rule to write $\delta\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\delta q_i \frac{\partial L}{\partial \dot{q}_i} \right) - \delta q_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i} \right)$;

$$\delta\mathcal{S} = \int_{t_1}^{t_2} \mathrm{d}t\ \delta q_i \left(\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \right) + \underbrace{\left[\delta q_i \frac{\partial L}{\partial q_i} \right]_{t_1}^{t_2}}_{\delta q_i(t_1) = \delta q_i(t_2) = 0}.$$

Therefore a particle taking the path of least action obeys the Euler-Lagrange equation of motion, i.e.

$$\frac{\delta\mathcal{S}}{\delta q_i} = \frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0$$

Proof 1.3: Euler-Lagrange equation of $L\left(q_i, \dot{q}_i, \ddot{q}_i\right)$

Following the principle of least action as in Proof 1.2, the variation in action is given by

$$\begin{aligned} \delta\mathcal{S} &= \int_{t_1}^{t_2} \delta L\left(q_i, \dot{q}_i, \ddot{q}_i\right)\ \mathrm{d}t \\ &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i + \frac{\partial L}{\partial \ddot{q}_i}\delta\ddot{q}_i \right)\ \mathrm{d}t. \end{aligned}$$

By the product rule, the derivatives become

$$\begin{aligned} \frac{\partial L}{\partial \dot{q}_i} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i \right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i} \right)\delta q_i \\ \frac{\partial L}{\partial \ddot{q}_i} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \ddot{q}_i}\delta\dot{q}_i \right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \ddot{q}_i} \right)\delta\dot{q}_i \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \ddot{q}_i}\delta\dot{q}_i \right) - \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \ddot{q}_i} \right)\delta\dot{q}_i \right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \ddot{q}_i} \right)\delta\dot{q}_i \right). \end{aligned}$$

As such, the variation in action is given by

$$\delta\mathcal{S} = \int_{t_1}^{t_2} \delta q_i \left(\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\right) \mathrm{d}t + \left[\frac{\partial L}{\partial \dot{q}_i}\delta q_i + \frac{\partial L}{\partial \ddot{q}_i}\delta \dot{q}_i - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\delta q_i\right]_{t_1}^{t_2}.$$

Using that $\delta q_i\left(t_{1,2}\right) = \delta \dot{q}_i\left(t_{1,2}\right) = 0$ and invoking the principle of least, i.e. $\delta\mathcal{S} = 0$, then one obtains the Euler-Lagrange equation as

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) = 0.$$

1.3 Lagrangian Field Theory

1.3.1 Classical Field Theory

In Lagrangian dynamics there is a finite number of degrees of freedom (DoF). For instance, a 3D system with N particles will usually have $3N$ DoF. The transition from finite to infinite degrees of freedom is necessary for continuous systems such as those found in quantum field theory, in whcih fields permeate all space. The continuous case can be approached by finding the correct limit.

Example 1.2: Approximating a rod as an infinite DoF mass-spring system

Consider an elastic rod of length ℓ undergoing small longitudinal vibrations of length ϕ. One may approximate the rod as a chain of n equal bodies of mass m connected by uniform massless springs of length a and constant k, as shown in Figure 1.2.

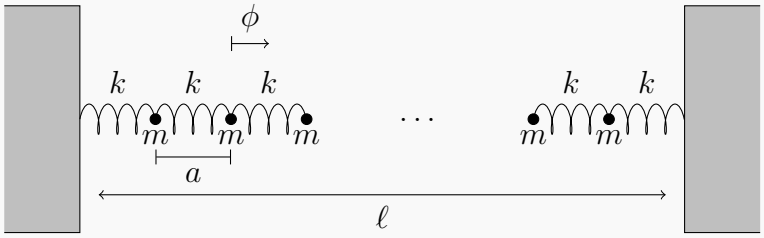


Figure 1.2: Representation of a rod as a system of masses on springs.

The total kinetic energy of the rod T is given by

$$T = \frac{1}{2}\sum_{i=1}^{n} m\dot{\phi}_i^2,$$

where ϕ_i is the displacement of the i^{th} mass from equilibrium. The total potential energy V is given by

$$V = \frac{1}{2}\sum_{i=0}^{n} k\left(\phi_{i+1} - \phi_i\right)^2.$$

The force on the i^{th} particle is then

$$F_i = \frac{\partial V}{\partial \phi_i} = k\left(\phi_{i+1} + \phi_{i-1} - 2\phi_i\right)$$

and the Lagrangian is

$$L = \frac{1}{2}\sum_{i=1}^{n} m\dot{\phi}_i^2 - \frac{1}{2}\sum_{i=0}^{n} k(\phi_{i+1} - \phi_i)^2.$$

To describe the original elastic rod, substitute for the total length $\ell = (n+1)a$, mass density $\mu = \frac{m}{L}$, and Young's modulus $Y = ka$. The Lagrangian then becomes

$$\begin{aligned} L &= \frac{1}{2}\sum_{i=1}^{n} \frac{a}{a} m\dot{\phi}_i^2 - \frac{1}{2}\sum_{i=0}^{n} \frac{a^2}{a^2} k(\phi_{i+1} - \phi_i)^2 \\ &= \frac{1}{2}\sum_{i=1}^{n} a\left(\frac{m}{a}\dot{\phi}_i^2\right) - \frac{1}{2}\sum_{i=0}^{n} a\,(ka)\left(\frac{\phi_{i+1} - \phi_i}{a}\right)^2. \end{aligned}$$

In the limit $n \to \infty$, then $a \to 0$ and hence $\frac{\phi_{i+1}-\phi_i}{a} \to \partial_\mu \phi$ from the definition of the derivative. Furthermore, within the limit $a \to 0$ the summation becomes an integral with $a \to \mathrm{d}x$, i.e.

$$L = \frac{1}{2} \int_0^\ell \mathrm{d}^3 x \ \left(\mu \dot{\phi}^2 - Y \left(\partial_x \phi \right)^2 \right).$$

The Lagrangian density is related to the Lagrangian as $L = \int \mathrm{d}^3 x \mathcal{L}$, thus is given by

$$\mathcal{L} = \frac{1}{2} \left(\mu \dot{\phi}^2 - Y \left(\partial_x \phi \right)^2 \right).$$

The transition of classical mechanics to field theory may be summarised as follows:

$$\begin{aligned} x(t) &\mapsto \phi(x,t) \\ \dot{x}(t) &\mapsto \dot{\phi}(x,t) \\ L(x,\dot{x}) &\mapsto \mathcal{L}[\phi, \dot{\phi}]. \end{aligned}$$

1.3.2 Relativistic Field Theory

In relativistic field theory the action is defined by

$$\mathcal{S}[\phi(x)] \coloneqq \int_{-\infty}^{\infty} \mathrm{d}^4 x \ \mathcal{L}(\phi(x), \partial_\mu \phi(x)),$$

where x denotes the position four-vector. Units of variables in field theory are measured as powers of energy and are of either length (E^{-1}) or mass (E^{+1}), e.g. $[t] = [x] = E^{-1}$; $[E] = [p] = E^{+1}$; $[\mathcal{S}] = E^0$; $[\phi] = E^{+1}$. The functional differentiation in this case is now

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \delta^{(4)}(x - y).$$

Proof 1.4: Deriving the equation of motion from Hamilton's principle of least action II

One may follow the same procedure as Proof 1.2, i.e. apply Hamilton's principle for the variation of the field and its derivatives.

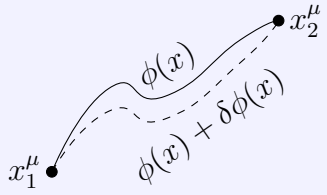


Figure 1.3: Possible paths taken by a particle in relativistic field theory.

The action for the path $\phi+\delta\phi$, again, requires a Taylor expansion. To first order in $\delta\phi$ this is

$$\begin{aligned}\delta\mathcal{S} &= \int_{-\infty}^{\infty} \mathrm{d}^4x \left(\frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\,\partial_\mu\delta\phi\right)\\ &= \int_{-\infty}^{\infty} \mathrm{d}^4x\;\delta\phi\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\right)\right) + \underbrace{\left[\delta\phi\,\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\right]_{-\infty}^{\infty}}_{\delta\phi(-\infty)=\delta\phi(\infty)=0},\end{aligned}$$

and hence the equation of motion for a relativistic particle obeys the relativistic Euler-Lagrange equation, i.e.

$$\frac{\delta\mathcal{S}}{\delta\phi} = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\right) = 0.$$

Referring back to Example 1.1, the Lagrangian density of a mass-sping system is

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.$$

This holds for 1+0-dimensional systems, however our universe is 1+3-dimensional. As such, the Lagrangian density is given by

$$\mathcal{L}[\phi(x)] = \frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - \frac{1}{2}m^2\phi^2.$$

The equation of motion of a relativistic mass-spring system is the Klein-Gordon equation, i.e.

$$\left(\Box + m^2\right)\phi = 0.$$

Proof 1.5: The equation of motion of a relativistic mass-spring system is the Klein-Gordon equation

The relativistic Euler-Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \right) = 0.$$

The derivatives are then

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi(x);$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} &= \frac{1}{2}\left(\frac{\partial\left(\partial_\nu \phi\right)}{\partial\left(\partial_\mu \phi\right)} \left(\partial^\nu \phi\right) + \left(\partial_\nu \phi\right) \frac{\partial\left(\partial^\nu \phi\right)}{\partial\left(\partial_\mu \phi\right)} \right) \\
&= \frac{1}{2}\left(\delta^\mu_\nu \partial^\nu \phi + \eta_{\nu\alpha}\partial^\alpha \eta^{\mu\nu}\right) \\
&= \frac{1}{2}\left(\delta^\mu_\nu \partial^\nu \phi + \delta^\mu_\alpha \partial^\alpha \phi\right) \\
&= \partial^\mu \phi.
\end{aligned}$$

The Euler-Lagrange equation of motion for a mass-spring system is therefore the Klein-Gordon equation, i.e.

$$\left(\partial_\mu \partial^\mu + m^2\right)\phi = \left(\Box + m^2\right)\phi = 0.$$

The Lorentz tranform is given by $x^\mu \mapsto x^{\mu\prime} = \Lambda^\mu_\nu x^\nu$, where $\Lambda^\mu_\nu \in \mathrm{SO}(1,3)$. One should also note that $\Lambda^\mu_\alpha \Lambda^\nu_\beta \eta^{\alpha\beta} = \eta^{\mu\nu}$.

The Lagrangian for electromagnetic field $F_{\mu\nu}$ describing a spin-1 particle with two physical degrees of freedom A^μ is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, i.e.

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}.$$

The components of the Euler-Lagrange equation of motion are

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = -J^\nu;$$
$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \big(\partial_\mu A_\nu\big)} \right) = -\partial_\mu F^{\mu\nu},$$

hence the Euler-Lagrange equation is

$$\partial_\mu F^{\mu\nu} = J^\nu = (\rho, \mathbf{J}).$$

1.4 Local and Global Symmetries

Previously the fields considered have only been real, i.e. $\phi^*(x) = \phi(x)$. If, instead, complex scalar fields are considered, i.e. $\phi^*(x) \neq \phi(x)$, then the Lagrangian density becomes

$$\mathcal{L} = (\partial^\mu \phi^*)\left(\partial_\mu \phi\right) - m^2 \phi^* \phi + \lambda \left(\phi^* \phi\right)^2.$$

Under a unitary rotation of angle θ, denoted U(1), the complex scalar fields transform as

$$\phi(x) \overset{\mathrm{U}(1)}{\longmapsto} \phi'(x) = e^{i\theta}\phi(x);$$
$$\phi^*(x) \overset{\mathrm{U}(1)}{\longmapsto} \phi^{*\prime}(x) = e^{-i\theta}\phi^*(x).$$

If θ has no x dependency, i.e. $\theta =$ constant, such a transformation is called a **global transformation**. However, if θ does have an x dependency, i.e. $\theta = \theta(x)$, the transformation is a **local** or **gauge transformation**. Under global rotation the Lagrangian density is invariant, i.e.

$$\mathcal{L}[\phi(x)] \mapsto \mathcal{L}[\phi'(x)] = \mathcal{L}[\phi(x)],$$

which is tantamount to saying that if the Universe were to be rotated through an angle θ then the Lagrangian density would be identical to that pre-transformation. However, under local transformations the Lagrangian density is not invariant, which is to say that if every point in the space were independently rotated through an angle θ the resulting Lagrangian density would be different.

Proof 1.6: Global and local U(1) transformations of the Lagrangian for a complex scalar field

Under a U(1) transformation, the Lagrangian transforms as

$$\begin{aligned}\mathcal{L} \xmapsto{\mathrm{U}(1)} \mathcal{L}' &= \left(\partial^\mu \phi^{*'}\right)\left(\partial_\mu \phi'\right) - m^2 \phi^{*'}\phi' + \lambda\left(\phi^{*'}\phi'\right)^2 \\ &= \left(\left(\partial^\mu e^{-i\theta}\right)\phi^* + e^{-i\theta}\left(\partial^\mu \phi^*\right)\right)\left(\left(\partial_\mu e^{i\theta}\right)\phi + e^{i\theta}\left(\partial_\mu \phi\right)\right) \\ &\qquad - m^2 e^{-i\theta}e^{i\theta}\phi^*\phi + \lambda\left(e^{-i\theta}e^{i\theta}\phi^*\phi\right)^2 \\ &= \mathcal{L} + \delta\mathcal{L},\end{aligned}$$

where $\delta\mathcal{L} = \left(\partial^\mu\theta\right)\left(\partial_\mu\theta\right)\phi^*\phi + i\left(\partial_\mu\theta\right)\left(\partial^\mu\phi^*\right)\phi - i\left(\partial^\mu\theta\right)\phi^*\left(\partial_\mu\phi\right)$. Under a global transformation, where $\partial^\mu\theta = \partial_\mu\theta = 0$, the derivatives vanish and so $\delta\mathcal{L} = 0$, thus the Lagrangian is invariant, i.e. $\mathcal{L} = \mathcal{L}'$. However, under a local (gauge) transformation the derivatives are non-zero and so the Lagrangian is not invariant as $\delta\mathcal{L} \neq 0$.

1.4.1 Infinitesimal Transformations

Infinitesimal transformations are given by

$$\phi_i(x) \mapsto \phi_i'(x) = \phi(x) + \delta\phi_i(x),$$

where

$$\delta\phi_i(x) = i\theta^a(x)\left(T^a\right)_i^j \phi_j,$$

where T^a are the generators of the Lie group, which is a group of smooth (infinitely differentiable) manifolds.

1.4.2 Special Matrices

Special Orthogonal 2×2 Matrices, SO(2)

The complex scalar fields may be written as a two-component, i.e.

$$\phi = \begin{pmatrix} \mathrm{Re}(\phi) \\ \mathrm{Im}(\phi) \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

SO(2) are real orthogonal matrices O belonging to the Lie groups with $O^T O = O O^T = \mathbb{I}$ and $\det(O) = 1$, otherwise written

$$\mathrm{SO}(2) = \left\{ O \in \mathrm{GL}(2, \mathbb{R}) \;\middle|\; O^T O = O O^T = \mathbb{I}_2 \cap \det(O) = 1 \right\}.$$

The Lie generator T is then

$$T = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = e^{i\sigma_2 \theta},$$

where $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, i.e.

$$\phi \xmapsto{\mathrm{SO}(2)} \phi' = \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

A property of SO(2) matrices is that

$$\phi_1^2 + \phi_2^2 = \phi_1'^2 + \phi_2'^2.$$

Special Unitary 2×2 Matrices, SU(2)

SU(2) are complex unitary matrices belonging to the Lie group, with $U^\dagger U = U U^\dagger = \mathbb{I}$ and $\det\{U\} = 1$, i.e.

$$\mathrm{SU}(2) = \left\{ U \in \mathrm{GL}(2, \mathbb{C}) \;\middle|\; U^T U = U U^T = \mathbb{I}_2 \cap \det(O) = 1 \right\}.$$

This means that

$$\phi \xmapsto{\text{SU(2)}} \phi' = \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = e^{i\theta^a T^a}\phi,$$

where $\theta^a \in \mathbb{R} : a = 1, 2, 3$ and $T^a = \frac{1}{2}\sigma_a$ are Lie generators, where σ_a are Pauli matrices, i.e.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}; \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

As a result, $\left[T^a, T^b\right] = i\epsilon_{abc}T^c$. A property of SU(2) matrices is that

$$|\phi_1|^2 + |\phi_2|^2 = \left|\phi_1'\right|^2 + \left|\phi_2'\right|^2.$$

1.4.3 Groups

A group G is a set of elements $\{a, b, c, \ldots\}$ which includes a "multiplcation" rule, such that if $a, b \in G$ then $(a \cdot b) \in G$ - this is known as the property of closure. If $ab = ba$ then the group is Abelian, however if $ab \neq ba$ then the group is non-Abelian. There are a number of other group properties:

Associativity	$(ab)c = a(bc)$	
Identity element	$\hat{\mathbf{e}}a = a\hat{\mathbf{e}} = \mathbf{a}$	$\hat{\mathbf{e}}$ is the identity of a
Inverse element	$a^{-1}a = aa^{-1} = \hat{\mathbf{e}}$	a^{-1} is the inverse of a
Order	The number of elements in G	

A representation of a group is a mapping that takes group elements $g \in G$ into linear operator F that preserve the composition rules of the group, i.e. $F(a)F(b) = F(ab)$. An element in a group G is specified by a finite set of n parameters:

$$g = G\{\theta_1, \theta_2, \ldots, \theta_n\}$$
$$\hat{\mathbf{e}} = G\{0, 0, \ldots, 0\}.$$

A Lie group is when the parameters are finite in number and continuous, and derivatives of the group elements with respect to their parameters exist.

The generators of a group are defined to be

$$x_i = -i \left. \frac{\partial g}{\partial \theta_i} \right|_{\theta_i} = 0.$$

The Lie algebra of a group is defined as the relationship of a group's commutators, i.e.

$$\left[x_i, x_j\right] = i f_{jkl} x_k,$$

where f_{jkl} is the structure function.

Some groups include:

Group	Definition	Example
O(N)	Orthogonal $N \times N$ matrices	
SO(N)	Orthogonal $N \times N$ matrices, det $= 1$	
SO(3)	Orthogonal 3×3 matrices, det $= 1$	3D rotations
U(N)	Unitary $N \times N$ matrices	
SU(N)	Unitary $N \times N$ matrices, det $= 1$	
SU(3)	Unitary $N \times N$ matrices, det $= 1$	Pauli Matrices

A Casimir operator is a non-linear function of the generators of a group, and it commutes with them. The number of C-operators (creation and annihilation) is the rank of the group - these are also invariant, i.e. $\hat{J}^2 = J_x^2 + J_y^2 + J_z^2$.

1.5 Noether's Theorem

If a Lagrangian density $\mathcal{L}$ is, up to a total derivative $\partial_\mu Q^\mu$, invariant under a given transformation of fields ϕ_i and spacetime x^μ then there is a conserved current $J^\mu(x)$, i.e. $\partial_\mu J^\mu = 0$, and a conserved charge Q, i.e. $\frac{\mathrm{d}Q}{\mathrm{d}t} = 0$, such that

$$Q = \int \mathrm{d}^3\mathbf{x}\ J^0(x).$$

Proof 1.7: Invariance under infinitesimal global transformations results in a conserved current and a conserved charge

Consider a transformation T^a that is infinitesimal, i.e. $a \ll 1$

$$\begin{aligned}\phi_i \xmapsto{\mathrm{SU}(N)} \phi_i' &= \left(e^{i\theta^a T^a}\right)_i^{\ j} \phi_j \\ &\approx \phi_i + \underbrace{i\theta^a \left(T^a\right)_i^{\ j}}_{=\delta\phi} + \mathcal{O}(\theta^2).\end{aligned}$$

This is such that the Lagrangian density transforms as

$$\begin{aligned}\mathcal{L}(\phi_i, \partial_\mu\phi_i) \mapsto \mathcal{L}(\phi_i', \partial_\mu\phi_i') &= \mathcal{L}(\phi_i', \partial_\mu\phi_i') \\ &= \mathcal{L}(\phi_i + \delta\phi_i, \partial_\mu(\phi_i + \delta\phi_i)) \\ &= \mathcal{L}(\phi_i, \partial_\mu\phi_i) + \underbrace{\delta\phi_i \frac{\partial\mathcal{L}}{\partial\phi_i} + \partial_\mu\left(\delta\phi_i\right) \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_i\right)}}_{=\delta\mathcal{L}}.\end{aligned}$$

For invariance, this must be equal to the initial Lagrangian density, hence $\delta\mathcal{L} \overset{!}{=} 0$. Therefore by the product rule (as earlier)

$$\begin{aligned}0 &= \delta\phi_i \frac{\partial\mathcal{L}}{\partial\phi_i} + \partial_\mu\left(\delta\phi_i\right) \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_i\right)} \\ &= \delta\phi_i \frac{\partial\mathcal{L}}{\partial\phi_i} + \partial_\mu\left(\delta\phi_i \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_i\right)}\right) - \delta\phi_i\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_i\right)}\right) \\ &= \partial_\mu \underbrace{\left(\delta\phi_i \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_i\right)}\right)}_{=A^\mu} + \delta\phi_i \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_i\right)}\right)\right)}_{=0 \text{ by Euler-Lagrange}},\end{aligned}$$

hence $\partial_\mu A^\mu = 0$. The result of this is that $A^\mu \propto J^\mu$, which is the expected form. As the infinitesimal variation in field is $\delta\phi_i = i\theta^a \left(T^a\right)_i^{\ j} \phi_j$, there are different conserved currents as the transformation $T^{a=1}$ is dif-

ferent to that of $T^{a=2}$. Therefore

$$\begin{aligned} J^{a,\mu} &= \frac{\partial \mathcal{L}}{\partial\big(\partial_\mu \phi_i\big)} \frac{\partial\left(\delta\phi_i\right)}{\partial(\theta^a)} \\ &= \frac{\partial \mathcal{L}}{\partial\big(\partial_\mu \phi_i\big)}\, i\left(T^a\right)_i^j \phi_j + \frac{\partial(\delta Q^\mu)}{\partial(\theta^a)}\,, \end{aligned}$$

with $Q^a(t) = \int \mathrm{d}^3x J^{a,\mu}$. This is conserved because $\partial_\mu J^\mu = 0 \to \partial_0 J^0 = -\boldsymbol{\nabla}\cdot\mathbf{J}$, therefore

$$\begin{aligned} \frac{\mathrm{d}Q^a}{\mathrm{d}t} &= \int \mathrm{d}^3\mathbf{x}\ \partial_0 J^{a,0} \\ &= -\int \mathrm{d}^3\mathbf{x}\ \boldsymbol{\nabla}\cdot\mathbf{J} \\ &= -\int \mathrm{d}\mathbf{S}\cdot\mathbf{J}, \end{aligned}$$

which is zero by Gauss' law. This is only possible when surface terms vanish at infinity, or requiring $\lim\limits_{x\to\pm\infty} \phi(x) = 0$.

Chapter 2

Canonical Quantisation

2.1 From Classical to Quantum Mechanics

2.1.1 Summary of Quantum Mechanics

1. Physical systems are ket vectors $|A\rangle$ in Hilbert space $\mathscr{H}$. A Hilbert space is an abstract vector space possessing the structure of an inner product that allows lengths and angles to be measured. The space is complete (infinite dimensional).

2. If a system is in state $|A\rangle$, the probability of it being observed in another state $|B\rangle$ is given by

$$P_{A\to B} = \left|\langle B|A\rangle\right|^2,$$

where

$$\begin{aligned}\langle B|A\rangle &= \int \langle B|x\rangle \langle x|A\rangle \ \mathrm{d}x \\ &= \int \Psi_B^*(x)\Psi_A(x)\ \mathrm{d}x.\end{aligned}$$

$\langle B|$ is a bra vector which is dual to the ket $|B\rangle$, i.e. $\langle B|^\dagger = |B\rangle$ or $\langle x|B\rangle^* = \langle B|x\rangle$.

3. Observables are mapped using Hermitian operators, i.e. $\hat{O} = \hat{O}^\dagger$.

4. Free particles are defined by plane waves, i.e.
$$\phi_{\mathbf{p}}(t,\mathbf{x}) = \langle t;\mathbf{x}|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}}e^{-ip_\mu x^\mu},$$
where $E_{\mathbf{p}} = \sqrt{m^2+p^2}$.

5. The Hilbert space is complete and orthogonal (completeness relation), i.e.
$$\sum_{\mathbf{p}} |\mathbf{p}\rangle \langle\mathbf{p}| = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3(2E_{\mathbf{p}})} |\mathbf{p}\rangle \langle\mathbf{p}| = \mathbb{I}$$
$$\langle\mathbf{k}|\mathbf{p}\rangle = (2\pi)^3\ 2E_{\mathbf{p}}\ \delta^{(3)}(\mathbf{k}-\mathbf{p}).$$

Proof 2.1: Deriving the Klein-Gordon from the momentum operator $\hat{P}_\mu$, which defined by $\hat{P}_\mu |\mathbf{x};t\rangle = i\dfrac{\partial}{\partial x^\mu}|\mathbf{x};t\rangle$

In special relativity, the four-momentum is defined by $p_\mu = (E,\mathbf{p})$, with its counterpart as $p^\mu = \eta^{\mu\nu}p_\nu = (E,-\mathbf{p})$. As such, the inner product of the four-momentum with itself is

$$p^\mu p_\mu = E^2 - p^2 = m^2.$$

The momentum operator defined by $\hat{P}_\mu = i\dfrac{\partial}{\partial x^\mu} = i\,(\partial_t, \boldsymbol{\nabla})$ has a covariant counterpart $\hat{P}^\mu = \eta^{\mu\nu}\hat{P}_\nu = i\,(\partial_t, -\boldsymbol{\nabla})$. As such, the combination of the operators is given by

$$\hat{P}^\mu\hat{P}_\mu = -\partial_t^2 + \nabla^2 \stackrel{!}{=} m^2.$$

When acting upon a state, the combined momentum operators hence give the Klein-Gordon equation:

$$\hat{P}^\mu\hat{P}_\mu |\mathbf{x};t\rangle = \left(-\partial_t^2+\nabla^2\right)|\mathbf{x};t\rangle = m^2|\mathbf{x};t\rangle$$
$$\Rightarrow \left(\partial_t^2 - \nabla^2 + m^2\right)|\mathbf{x};t\rangle = 0.$$

2.1.2 Physical States

Physical states are tensors in **Fock space** $\mathcal{F}$.

Definition 2.1: Fock Space

A Fock space is a framework used in quantum field theory to construct quantum states of an unknown quantity of particles. More formally, a Fock space consists of a set of Hilbert spaces representing n-particle states, i.e. 0-particle, 1-particle, 2-particle... states.

A 0-particle state, denoted $|0;t_0\rangle$, represents a vacuum state at a time t_0. A state with a single particle of momentum $\mathbf{p}$ at time t_0 is instead denoted $|\mathbf{p};t_0\rangle$ and with a second particle of momentum $\mathbf{q}$ is $|\mathbf{p},\mathbf{q};t_0\rangle$.

If bosons are being considered, the n-particle state is a symmetrised tensor product of n 1-particle Hilbert spaces, however if fermions are being considered then the n-particle state is an antisymmetric tensor product of n 1-particle Hilbert spaces. The Fock space $\mathcal{F}$ is the direct sum of these symmetric or antisymmetric tensors of the single particle Hilbert spaces, i.e.

$$\mathcal{F} = \sum_{i_n=0}^{\infty} S_0\mathscr{H}_{i_0} \oplus S_1\mathscr{H}_{i_1} \oplus \ldots \oplus S_n\mathscr{H}_{i_n} = \bigoplus_{i_n=0}^{\infty} S_{i_n}\mathscr{H}_{i_n},$$

where S_n is the operator that either symmetrises or antisymmetrises the tensor, depending on whether the particle is bosonic or fermionic.

2.1.3 Creation and Annihilation Operators

The creation operator $\hat{a}^\dagger(\mathbf{p})$ adds a particle with momentum $\mathbf{p}$ to the Fock space, i.e.

$$\hat{a}^\dagger(\mathbf{p})\,|\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_n\rangle = |\mathbf{p},\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_n\rangle\,.$$

The creation operator's adjoint, the annihilation operator $\hat{a}(\mathbf{p})$, removes a particle of momentum $\mathbf{p}$ from the state on which it acts, as such

$$\hat{a}(\mathbf{p})\,|\mathbf{p}\rangle \equiv |0\rangle\,.$$

The annihilation operator works by producing an antiparticle to annihilate with the original particle, i.e.

$$\begin{aligned}\hat{a}(\mathbf{p})\left|\mathbf{p}'\right\rangle &= \left\langle 0\right|\hat{a}(\mathbf{p})\left|\mathbf{p}'\right\rangle \\ &= \left\langle \mathbf{p}'\right|\hat{a}^\dagger(\mathbf{p})\left|0\right\rangle^* \\ &= \left\langle \mathbf{p}'\right|\left.\mathbf{p}\right\rangle \\ &= (2\pi)^3\ 2E_{\mathbf{p}}\ \delta^{(3)}(\mathbf{p}-\mathbf{p}'),\end{aligned}$$

where the latter result derives from the commutation relation between the creation and annihilation operators, i.e.

$$\begin{aligned}\left[\hat{a}(\mathbf{p}),\hat{a}^\dagger(\mathbf{q})\right] &= \left\langle 0\right|\left[\hat{a}(\mathbf{p}),\hat{a}^\dagger(\mathbf{q})\right]\left|0\right\rangle \\ &= \left\langle 0\right|\hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{q})\left|0\right\rangle - \left\langle 0\right|\hat{a}^\dagger(\mathbf{q})\hat{a}(\mathbf{p})\left|0\right\rangle \\ &= \left\langle 0\right|\hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{q})\left|0\right\rangle - 0 \\ &= \left\langle \mathbf{q}\right|\hat{a}^\dagger(\mathbf{p})\left|0\right\rangle^* \\ &= \left\langle \mathbf{p}\right|\left.\mathbf{q}\right\rangle \\ &= (2\pi)^3\ 2E_{\mathbf{p}}\ \delta^{(3)}(\mathbf{p}-\mathbf{q}).\end{aligned}$$

2.2 Quantum Fields and Causality

2.2.1 Field Operator

Consider the Klein-Gordon equation;

$$\left(\Box + m^2\right)\phi(x) = 0,$$

where $\Box = \partial_\mu\partial^\mu$. For a real field, i.e. $\phi = \phi^*$, one can ansatz a plane wave solution with four-momentum k_μ, i.e.

$$\phi(x) \propto \exp\left(i\left(k_0 t - \mathbf{k}\cdot\mathbf{x}\right)\right),$$

where $k_0 = E$. The Klein-Gordon equation then becomes

$$\left(-k_0^2 + k^2 + m^2\right)\phi(x) = 0,$$

which gives energy eigenvalues $E = \pm\left(k^2 + m^2\right)^{\frac{1}{2}}$. Taking the positive solution, one can define the energy as

$$E_k = E(k) = \left(k^2 + m^2\right)^{\frac{1}{2}} \geq 0.$$

The scalar field then has two solutions:

$$\phi_{\pm} \propto \exp\left(\pm i\left(E_k - \mathbf{k}\cdot\mathbf{x}\right)\right).$$

The general solution is a superposition of both solutions, i.e.

$$\phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3\ 2E_k}\left(a^*(k)e^{i\mathbf{k}\cdot\mathbf{x}} + a(k)e^{-i\mathbf{k}\cdot\mathbf{x}}\right),$$

where $a(k)$ and $a^*(k)$ are arbitrary Fourier coefficients.

After quantisation, the Klein-Gordon equation becomes an equation for operators. In classical field theory, $\phi(x)$ plays the role of coordinates, continuing the analogy to point-like particle mechanics. One takes the canonical variables to be operators:

$$\begin{aligned}\phi(x) &\mapsto \hat{\phi}(x)\\ \pi(x) &\mapsto \hat{\pi}(x),\end{aligned}$$

where $\pi(x)$ is the conjugate momentum, which is defined as

$$\pi \coloneqq \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial x}{\partial t}\right)}.$$

By analogy, complex conjugates become Hermitian conjugates, i.e. $* \mapsto \dagger$, and so the scalar field is now

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3\ 2E_k}\left(a^\dagger(k)e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}(k)e^{-i\mathbf{k}\cdot\mathbf{x}}\right).$$

Before proceeding it worth recapping some commutation relations:

1. $\left[\hat{x}_i(t), \hat{x}_j(t)\right] = 0$
2. $\left[\hat{x}_i(t_1), \hat{x}_j(t_2)\right] \neq 0$
3. $\left[\hat{x}_i(t), \hat{p}_j(t)\right] = i\delta_{ij}$.

Proof 2.2: Identities

1.

$$\begin{aligned}\left[\hat{x}_i(t), \hat{x}_j(t)\right] &= x_i(t) \cdot x_j(t) - x_j(t) \cdot x_i(t) \\ &= \delta_{ij} - \delta_{ji} \\ &= 0.\end{aligned}$$

2.

$$\begin{aligned}\left[\hat{x}_i(t_1), \hat{x}_j(t_2)\right] &= x_i(t_1) \cdot x_j(t_2) - x_j(t_2) \cdot x_i(t_1) \\ &= \delta_{ij}\delta(t_1 - t_2) - \delta_{ji}\delta(t_2 - t_1) \\ &= 2\delta_{ij}\delta(t_1 - t_2).\end{aligned}$$

3.

$$\begin{aligned}\left[\hat{x}_i(t), \hat{p}_j(t)\right] |f\rangle &= x_i(t) \cdot p_j(t) |f\rangle - p_j(t) \cdot \left(x_i(t) |f\rangle\right) \\ &= -\left(p_j(t) \cdot x_i(t)\right) |f\rangle \\ &= i \frac{\partial x_i}{\partial x_j} |f\rangle \\ &= i\delta_{ij} |f\rangle .\end{aligned}$$

To solve for $\hat{a}$ and $\hat{a}^\dagger$ in terms of $\hat{\phi}$, one inverts the Fourier integral and its time derivative, i.e.

$$\begin{aligned}\int \mathrm{d}^3x \ \hat{\phi}(t, \mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} &= \frac{1}{2E}\left(\hat{a}(k) + \hat{a}^\dagger e^{2ik_0x_0}\right) \\ \int \mathrm{d}^3x \ \hat{\pi}(t, \mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} &= -\frac{i}{2}\left(\hat{a}(k) - \hat{a}^\dagger e^{2ik_0x_0}\right).\end{aligned}$$

Taking linear combinations of these give the creation and annihilation operators as

$$\begin{aligned}\hat{a}(k) &= \int \mathrm{d}^3x \left(E_k\hat{\phi}(t, \mathbf{x}) + i\hat{\pi}(t, \mathbf{x})\right) e^{i\mathbf{k}\cdot\mathbf{x}} \\ \hat{a}^\dagger(k) &= \int \mathrm{d}^3x \left(E_k\hat{\phi}(t, \mathbf{x}) - i\hat{\pi}(t, \mathbf{x})\right) e^{-i\mathbf{k}\cdot\mathbf{x}}.\end{aligned}$$

2.2.2 The Quantum Field Canonical Quantisation

For a free scalar field, i.e. one which satisfies the Klein-Gordon equation,

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3\ 2E_k} \left(a^\dagger(k) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}(k) e^{-i\mathbf{k}\cdot\mathbf{x}} \right)$$

with $\left[\hat{a}(\mathbf{p}), \hat{a}^\dagger(\mathbf{q})\right] = (2\pi)^3\ 2E_{\mathbf{p}}\ \delta^{(3)}(\mathbf{p}-\mathbf{q})$. In keeping with point mechanics, one may define conjugate momenta by analogy:

$$p_\mu = \frac{\partial \mathcal{L}}{\partial(\dot{x}^\mu)} \mapsto \hat{\pi} = \frac{\partial \mathcal{L}}{\partial(\dot{\phi}(x))}\,,$$

which is equal to $\partial_0 \phi$ for free scalar fields. This then gives the conjugate momentum as

$$\hat{\pi}(t, \mathbf{x}) = \int \frac{\mathrm{d}^3 \mathbf{p}}{(2\pi)^3\ 2E_{\mathbf{p}}} \frac{i}{2} \left(a^\dagger(p) e^{-i\mathbf{p}\cdot\mathbf{x}} - \hat{a}(p) e^{i\mathbf{p}\cdot\mathbf{x}} \right).$$

Furthermore, one can define the Hamiltonian function H and density $\mathscr{H}$ by analogy:

$$H(t) = \int \mathrm{d}^3 x\ \mathscr{H}\left[\phi, \pi\right],$$

where the Hamiltonian density is defined by $\mathscr{H}[\phi, \pi] = \pi\dot{\phi} - \mathcal{L}$.

For future convenience, a new notation will be introduced:

$$\int \frac{\mathrm{d}^3 \mathbf{p}}{(2\pi)^3\ 2E_{\mathbf{p}}} \mapsto \int_{\mathbf{p}}$$
$$\hat{a}(\mathbf{p}) \mapsto a_{\mathbf{p}}$$
$$\hat{a}^\dagger(\mathbf{p}) \mapsto a^\dagger_{\mathbf{p}}.$$

Analogous to the position-momentum commutation relations in quantum mechanics, the field theory conjugate position and momentum commutations are

$$\left[\hat{x}_i, \hat{p}_j\right] = i\delta_{ij} \mapsto \left[\hat{\phi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{y})\right] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$$
$$\left[\hat{x}_i, \hat{x}_j\right] = \left[\hat{p}_i, \hat{p}_j\right] = 0 \mapsto \left[\hat{\phi}(t, \mathbf{x}), \hat{\phi}(t, \mathbf{y})\right] = \left[\hat{\pi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{y})\right] = 0.$$

Proof 2.3: Conjugate position and momentum commutations

Using the earlier commutation relations $\left[a_{\mathbf{p}}, a^{\dagger}_{\mathbf{q}}\right] = (2\pi)^3\, 2E_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{q})$ and $\left[a^{\dagger}_{\mathbf{p}}, a_{\mathbf{q}}\right] = -(2\pi)^3\, 2E_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{q})$ with the Fourier representation of the delta function $\int_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} = \delta^3(\mathbf{x}-\mathbf{y})$, then

$$\begin{aligned}
\left[\hat{\phi}(t,\mathbf{x}), \hat{\pi}(t,\mathbf{y})\right] &= \int_{\mathbf{p}}\int_{\mathbf{q}} \left[a_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} + a^{\dagger}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}, \frac{i}{2}\left(a^{\dagger}_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{y}} - a_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{y}}\right)\right] \\
&= \frac{i}{2}\int_{\mathbf{p}}\int_{\mathbf{q}} \left[a_{\mathbf{p}}, a^{\dagger}_{\mathbf{q}}\right] e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} - \left[a^{\dagger}_{\mathbf{p}}, a_{\mathbf{q}}\right] e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \\
&= \frac{i}{2}\int_{\mathbf{p}} e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \\
&= \frac{i}{2}\left(\delta^{(3)}(\mathbf{y}-\mathbf{x}) + \delta^{(3)}(\mathbf{x}-\mathbf{y})\right) \\
&= i\delta^{(3)}(\mathbf{x}-\mathbf{y});
\end{aligned}$$

$$\begin{aligned}
\left[\hat{\phi}(t,\mathbf{x}), \hat{\phi}(t,\mathbf{y})\right] &= \int_{\mathbf{p}}\int_{\mathbf{q}} \left[a_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} + a^{\dagger}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}, a_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{y}} + a^{\dagger}_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{y}}\right] \\
&= \int_{\mathbf{p}}\int_{\mathbf{q}} \left[a_{\mathbf{p}}, a^{\dagger}_{\mathbf{q}}\right] e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} + \left[a^{\dagger}_{\mathbf{p}}, a_{\mathbf{q}}\right] e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \\
&= \int_{\mathbf{p}} e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \\
&= \delta^{(3)}(\mathbf{y}-\mathbf{x}) - \delta^{(3)}(\mathbf{x}-\mathbf{y}) \\
&= 0;
\end{aligned}$$

$$
\begin{aligned}
\left[\hat{\pi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{y})\right] &= \int_{\mathbf{p}}\int_{\mathbf{q}}\left[\frac{i}{2}\left(a_{\mathbf{p}}^{\dagger}e^{i\mathbf{p}\cdot\mathbf{x}}-a_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right),\frac{i}{2}\left(a_{\mathbf{q}}^{\dagger}e^{i\mathbf{q}\cdot\mathbf{y}}-a_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{y}}\right)\right] \\
&= \frac{1}{4}\int_{\mathbf{p}}\int_{\mathbf{q}}\left[a_{\mathbf{p}},a_{\mathbf{q}}^{\dagger}\right]e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}+\left[a_{\mathbf{p}}^{\dagger},a_{\mathbf{q}}\right]e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \\
&= \frac{1}{4}\int_{\mathbf{p}}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}-e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \\
&= \frac{1}{4}\left(\delta^{(3)}(\mathbf{y}-\mathbf{x})-\delta^{(3)}(\mathbf{x}-\mathbf{y})\right) \\
&= 0.
\end{aligned}
$$

2.2.3 Causality of the Field Operator

So far only equal-time commutations have been considered, however one might also wish to consider time as part of a four-vector commutation. For two events at spacetime coordinates x^{μ} and y^{μ}, the separation is

$$
\begin{aligned}
&\text{Space-like} \quad (x-y)^2 < 0 \\
&\text{Light-like} \quad (x-y)^2 = 0 \\
&\text{Time-like} \quad (x-y)^2 > 0.
\end{aligned}
$$

In non-relativistic quantum mechanics, if two operators do not commute, i.e. $\left[\hat{A},\hat{B}\right]\neq 0$, then one cannot measure A and B together perfectly. Therefore in quantum field theory, fields with space-like separations, the commutator must be equal to zero as the two locations are causally independent, i.e. information cannot travel between the points faster than the speed of light. This is mathematically written

$$
\left[\hat{\phi}(x),\hat{\phi}(y)\right]=\left[\hat{\phi}(x),\hat{\pi}(y)\right]=\left[\hat{\pi}(x),\hat{\pi}(y)\right]=0
$$

for $(x-y)^2<0$. The Pauli-Jordan function $\Delta(x-y)$, also known as the causal propagator or advanced-minus-retarded fundamental solution, is defined as

$$
\Delta(x-y) \coloneqq G_{\text{adv}}(x-y)-G_{\text{ret}}(x-y),
$$

where

$$G_{\text{adv}}(x-y) = -i\int_{\mathbf{p}} e^{-ip^{\mu}\left(x_{\mu}-y_{\mu}\right)}$$
$$G_{\text{ret}}(x-y) = -i\int_{\mathbf{p}} e^{ip^{\mu}\left(x_{\mu}-y_{\mu}\right)}$$

are the advanced and retarded Green's functions. The Pauli-Jordan function is Lorentz invariant and has the property $\Delta(\mathbf{x}-\mathbf{y}) = -\Delta(\mathbf{y}-\mathbf{x})$.

Proof 2.4: $\Delta(\mathbf{x}-\mathbf{y}) = -\Delta(\mathbf{y}-\mathbf{x})$

From Proof 2.3:

$$\begin{aligned}
\left[\hat{\phi}(t,\mathbf{x}),\hat{\phi}(t,\mathbf{y})\right] &= \int_{\mathbf{p}}\int_{\mathbf{q}}\left[a_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}} + a^{\dagger}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}, a_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{y}} + a^{\dagger}_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{y}}\right] \\
&= \int_{\mathbf{p}}\int_{\mathbf{q}}\left[a_{\mathbf{p}}, a^{\dagger}_{\mathbf{q}}\right]e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} + \left[a^{\dagger}_{\mathbf{p}}, a_{\mathbf{q}}\right]e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \\
&= \int_{\mathbf{p}} e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} = -i\Delta(\mathbf{y}-\mathbf{x}) \\
&= i\left(G_{\text{adv}}(\mathbf{x}-\mathbf{y}) - G_{\text{ret}}(\mathbf{x}-\mathbf{y})\right) \\
&= i\Delta(\mathbf{x}-\mathbf{y}).
\end{aligned}$$

One now may now equate the terms in red and hence derive the identity

$$\Delta(\mathbf{x}-\mathbf{y}) = -\Delta(\mathbf{y}-\mathbf{x}).$$

Consider two events x and y that occur at the same time but at different points in space, i.e. $x_0 = y_0$ but $x \neq y$. Defining the spacetime separation $z = x - y$, one can prove that these events are space-like separated:

$$\begin{aligned}
z^2 &= (x-y)^2 \\
&= (x_0-y_0)^2 - (\mathbf{x}-\mathbf{y})^2 \\
&= -(\mathbf{x}-\mathbf{y})^2 < 0 \Rightarrow \text{space-like.}
\end{aligned}$$

In order to prove the earlier assumption that equal-time commutators must be equal to zero, one can integrate the Pauli-Jordan function over all space,

i.e.

$$\begin{aligned}\Delta(\mathbf{x}-\mathbf{y}) &= \int_{-\infty}^{\infty} \underbrace{\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3\ 2E_{\mathbf{p}}}}_{\substack{\text{symmetric}\\ \Rightarrow \text{ even}}} \underbrace{\left(e^{-i\mathbf{p}\cdot\mathbf{z}} - e^{i\mathbf{p}\cdot\mathbf{z}}\right)}_{\substack{\text{anti-symmetric}\\ \Rightarrow \text{ odd}}} \\ &= \int_{-\infty}^{\infty} (\text{odd function})\, \mathrm{d}^3\mathbf{p} \\ &= 0.\end{aligned}$$

2.3 Canonical Quantisation of Scalar Field Theory

By this point classical mechanics has been pushed to quantum mechanics, then onto relativistic quantum mechanics and will soon be pushed further into the realm of quantum field theory. Relationships that could have been drawn by analogy have been proved, i.e.

- Classical mechanics

$$\begin{aligned}\{p_i, H\} = \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ \{q_i, H\} = \dot{q}_i &= -\frac{\partial H}{\partial p_i} \\ \Rightarrow \{q_i, p_i\} &= \delta_{ij}.\end{aligned}$$

- Quantum mechanics

$$\begin{aligned}\{\hat{q}, \hat{p}\} = -i\,[\hat{q}, \hat{p}] &= 1 \\ \Rightarrow [\hat{q}, \hat{p}] &= i.\end{aligned}$$

- Relativistic quantum mechanics

$$\left[\hat{x}^\mu(\tau), \hat{p}^\nu(\tau)\right] = -i\eta^{\mu\nu},$$

  where τ is the proper time.

Proof 2.5

Using the identity for the momentum operator $\hat{p}^\nu = i\partial^\nu$, then

$$\begin{aligned}\left[\hat{x}^\mu(\tau), \hat{p}^\nu(\tau)\right] |f\rangle &= x^\mu i\partial^\nu |f\rangle - i\partial^\nu \left(x^\mu |f\rangle\right) \\ &= -i\left(\partial^\nu x^\mu\right) |f\rangle \\ &= -i\left(\eta^{\mu\nu}\partial_\mu x^\mu\right) |f\rangle \\ &= -i\eta^{\mu\nu} |f\rangle .\end{aligned}$$

- Quantum field theory

$$\left[\hat{\phi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{y})\right] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}).$$

To quantise a theory for bosons, proceed as follows:

1. Write down the Lagrangian describing the system

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - \frac{1}{2}m^2\phi^2.$$

2. Promote classical fields to operators which satisfy equal-time commutators

$$\begin{aligned}\left[\hat{\phi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{y})\right] &= i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \\ \left[\hat{\phi}(t, \mathbf{x}), \hat{\phi}(t, \mathbf{y})\right] = \left[\hat{\pi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{y})\right] &= 0.\end{aligned}$$

3. Calculate the Hamiltonian density

$$\mathscr{H}(\phi, \pi) \coloneqq \pi(x)\partial_0\phi(x) - \mathcal{L}; \qquad \pi(x) = \partial_0\phi(x)$$

$$\begin{aligned}\Rightarrow \mathscr{H} &= (\partial_0\phi)^2 - \left(\frac{1}{2}\underbrace{\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right)}_{=(\partial_0\phi)^2 - (\boldsymbol{\nabla}\phi)^2} - \frac{1}{2}m^2\phi^2\right) \\ &= \frac{1}{2}\pi^2 + \frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2 + \frac{1}{2}m^2\phi^2. \qquad \left(\Rightarrow \langle\Psi | \mathscr{H} | \Psi\rangle \geq 0\right)\end{aligned}$$

2.4 Complex Fields and Antiparticles

2.4.1 Complex Fields

Consider now the case in which the field operators are complex, i.e. $\phi \neq \phi^\dagger$. The Lagrangian in such a case is

$$\mathcal{L} = \left(\partial_\mu \phi\right)\left(\partial^\mu \phi^\dagger\right) - m^2 \phi \phi^\dagger.$$

The conjugate momenta $\hat{\pi}$ are then

$$\hat{\pi} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial^0 \phi^\dagger$$
$$\hat{\pi}^\dagger = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \phi^\dagger\right)} = \partial^0 \phi.$$

As the conjugates of the complex field operators cannot be equal, a new set of creation and annihilation operators $\hat{b}$ must be introduced:

$$\phi(\mathbf{x}) = \int_{\mathbf{k}} a_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} + b_{\mathbf{k}}^\dagger e^{i\mathbf{k}\cdot\mathbf{x}}$$
$$\phi^\dagger(\mathbf{x}) = \int_{\mathbf{k}} a_{\mathbf{k}}^\dagger e^{i\mathbf{k}\cdot\mathbf{x}} + b_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}.$$

As before, one may use the commutation relations

$$\left[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger\right] = (2\pi)^3\ 2E_{\mathbf{k}}\ \delta^{(3)}(\mathbf{k} - \mathbf{k}')$$
$$\left[b_{\mathbf{k}}, b_{\mathbf{k}'}^\dagger\right] = (2\pi)^3\ 2E_{\mathbf{k}}\ \delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

to show that

$$\left[\hat{\phi}(t,\mathbf{x}), \hat{\pi}(t,\mathbf{y})\right] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$$
$$\left[\hat{\phi}^\dagger(t,\mathbf{x}), \hat{\pi}^\dagger(t,\mathbf{y})\right] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$$

while all other equal-time commutators vanish, i.e.

$$\left[\hat{\phi}(t,\mathbf{x}), \hat{\pi}^\dagger(t,\mathbf{y})\right] = \left[\hat{\phi}^\dagger(t,\mathbf{x}), \hat{\pi}(t,\mathbf{y})\right] = 0;$$
$$\left[\hat{\phi}(t,\mathbf{x}), \hat{\phi}^\dagger(t,\mathbf{y})\right] = \left[\hat{\pi}(t,\mathbf{x}), \hat{\pi}^\dagger(t,\mathbf{y})\right] = 0.$$

2.4.2 Conserved Currents

It has been seen that the Lagrangian for complex fields is given by

$$\mathcal{L} = \left(\partial_\mu \phi\right)\left(\partial^\mu \phi^\dagger\right) - m^2 \phi \phi^\dagger,$$

which is invariant under U(1) transformation.

Proof 2.6: Invariance of complex scalar field Lagrangian density under U(1) rotation

A U(1) transformation of a complex field is given by

$$\begin{aligned} \mathrm{U}(1) : \phi \mapsto \phi' = e^{-i\theta}\phi; \qquad & \delta\phi = -i\theta\phi \\ \mathrm{U}(1) : \phi^\dagger \mapsto \phi^{\dagger\prime} = e^{i\theta}\phi^\dagger; \qquad & \delta\phi^\dagger = i\theta\phi^\dagger. \end{aligned}$$

Therefore the Lagrangian under a U(1) transformation is

$$\begin{aligned} \mathrm{U}(1) : \mathcal{L} \mapsto \mathcal{L}' &= \left(\partial_\mu \phi^{\dagger\prime}\right)\left(\partial^\mu \phi'\right) - m^2 \phi^{\dagger\prime}\phi' \\ &= \left(\partial_\mu \left(e^{i\theta}\phi^\dagger\right)\right)\left(\partial^\mu \left(e^{-i\theta}\phi\right)\right) - m^2\left(e^{i\theta}\phi^\dagger\right)\left(e^{-i\theta}\phi\right) \\ &= \left(\partial_\mu \phi^\dagger\right)\left(\partial^\mu \phi\right) - m^2 \phi^\dagger \phi \\ &= \mathcal{L}. \end{aligned}$$

The conserved four-current J^μ can be derived from the Lagrangian to be

$$J^\mu = \phi^\dagger \left(i\partial^\mu \phi\right) - \left(i\partial^\mu \phi^\dagger\right)\phi.$$

Proof 2.7: Conserved four-current

The four-current under U(1) rotation is defined by the change in the

Lagrangian density under change in angle, i.e. $J^\mu = \frac{\partial \mathcal{L}}{\partial \theta}$, which is then

$$\begin{aligned} J^\mu &= \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \frac{\partial\left(\delta\phi\right)}{\partial\theta} + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^\dagger\right)} \frac{\partial\left(\delta\phi^\dagger\right)}{\partial\theta} \\ &= \left(\partial^\mu \phi^\dagger\right)\left(-i\phi\right) + \left(\partial^\mu \phi\right)\left(i\phi^\dagger\right) \\ &= \phi^\dagger\left(i\partial^\mu \phi\right) - \left(i\partial^\mu \phi^\dagger\right)\phi. \end{aligned}$$

2.4.3 Wick Ordering

Definition 2.2: Wick Ordering

Wick ordering, or normal ordering, is when all creation operators are to the left of all annihilation operators, i.e.

$$:\hat{a}\hat{a}^\dagger: = \hat{a}^\dagger \hat{a},$$

which removes vacuum effects.

The conserved charge Q is given by the spatial integral of the temporal conserved current term, i.e.

$$Q = \int_{-\infty}^{\infty} \mathrm{d}^3\mathbf{x} \;\; :J^0:,$$

where $:J^0:$ is the **Wick ordering** of the zeroth component of J. The conserved charge operator $\hat{Q}$ is then defined as

$$\hat{Q} \coloneqq Q - \langle 0 \,|\, Q \,|\, 0 \rangle \,.$$

Proof 2.8: Conserved charge operator

From the definition of the conserved four-current, the conserved charge

is

$$Q = \int_{-\infty}^{\infty} \mathrm{d}^3\mathbf{x} \ :J^0: \\ = \int_{-\infty}^{\infty} \mathrm{d}^3\mathbf{x} \ \left(\phi^\dagger \left(i\partial^0\phi\right) - \left(i\partial^0\phi^\dagger\right)\phi\right)$$

- complete

Therefore, by redefining the conserved charge with normal order, the issues of vacuum effects are removed, i.e.

$$Q = \int_{\mathbf{k}} \left(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} - b^\dagger_{\mathbf{k}} b_{\mathbf{k}}\right),$$

which as expected gives $\frac{\partial Q}{\partial t} = 0$.

2.4.4 More On Charge

By applying the conserved charge operator to a particle (denoted as a creation operator on a vacuum state), one can see that $a^\dagger_{\mathbf{k}}$ creates a particle of charge $+1$ and momentum $\mathbf{k}$, whereas $b^\dagger_{\mathbf{k}}$ creates an antiparticle of charge -1 and momentum $\mathbf{k}$.

Proof 2.9: Applications of $\hat{Q}$

Using the relations $a_{\mathbf{k}}a_{\mathbf{p}}^{\dagger} = a_{\mathbf{p}}^{\dagger}a_{\mathbf{k}} + (2\pi)^3 2E_{\mathbf{p}}\delta^{(3)}(\mathbf{k}-\mathbf{p})$ and $b_{\mathbf{k}}b_{\mathbf{p}}^{\dagger} = b_{\mathbf{p}}^{\dagger}b_{\mathbf{k}} + (2\pi)^3 2E_{\mathbf{p}}\delta^{(3)}(\mathbf{k}-\mathbf{p})$, then

$$\begin{aligned}
\hat{Q}a_{\mathbf{p}}^{\dagger}|0\rangle &= \int_{\mathbf{k}} \left(a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}} - b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right) a_{\mathbf{p}}^{\dagger}|0\rangle \\
&= \int_{\mathbf{k}} a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}a_{\mathbf{p}}^{\dagger}|0\rangle - \underbrace{\int_{\mathbf{k}} b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}a_{\mathbf{p}}^{\dagger}|0\rangle}_{=0} \\
&= \int_{\mathbf{k}} a_{\mathbf{k}}^{\dagger}a_{\mathbf{p}}^{\dagger} \underbrace{a_{\mathbf{k}}|0\rangle}_{\substack{\text{annihilates}\\ \text{vacuum}\\ \Rightarrow 0}} + \int_{\mathbf{k}} a_{\mathbf{k}}^{\dagger}(2\pi)^3 2E_{\mathbf{p}}\delta^{(3)}(\mathbf{k}-\mathbf{p})|0\rangle \\
&= a_{\mathbf{p}}^{\dagger}|0\rangle ;
\end{aligned}$$

$$\begin{aligned}
\hat{Q}b_{\mathbf{p}}^{\dagger}|0\rangle &= \int_{\mathbf{k}} \left(a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}} - b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right) b_{\mathbf{p}}^{\dagger}|0\rangle \\
&= \underbrace{\int_{\mathbf{k}} a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}b_{\mathbf{p}}^{\dagger}|0\rangle}_{=0} - \int_{\mathbf{k}} b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}b_{\mathbf{p}}^{\dagger}|0\rangle \\
&= -\int_{\mathbf{k}} b_{\mathbf{k}}^{\dagger}b_{\mathbf{p}}^{\dagger} \underbrace{b_{\mathbf{k}}|0\rangle}_{\substack{\text{annihilates}\\ \text{vacuum}\\ \Rightarrow 0}} - \int_{\mathbf{k}} b_{\mathbf{k}}^{\dagger}(2\pi)^3 2E_{\mathbf{p}}\delta^{(3)}(\mathbf{k}-\mathbf{p})|0\rangle \\
&= -b_{\mathbf{p}}^{\dagger}|0\rangle .
\end{aligned}$$

Chapter 3

The S-Matrix

3.1 Time Evolution of Quantum States and the S-Matrix

The dynamics of quantum mechanics can be mathematically formulated in multiple equivalent ways, known as interpretations. The Copenhagen interpretation of quantum mechanics has three formulations which differ in how certain objects vary with time; these are the Schrödinger, Heisenberg, and Dirac pictures.

The Hamiltonian of a system H may be split into a free (exact) Hamiltonian H_0 and an interaction (potential) Hamiltonian H_{int}, i.e.

$$H = H_0 + H_{\text{int}},$$

which is proportional to $\lambda\phi^4(x)$ in the Lagrangian density $\mathcal{L}$.

3.1.1 The Schrödinger Picture

In the Schrödinger picture (SP), quantum states are time-dependent whereas operators are time-independent. States evolve in time through a time-evolution operator $U(t, t_0)$. This operator acts upon a ket at a time t_0 to produce a

ket at some later time t, i.e.

$$\left|\Psi(t)\right\rangle = U(t,t_0)\left|\Psi(t_0)\right\rangle.$$

The equivalent equation for bras is

$$\left\langle\Psi(t)\right| = \left\langle\Psi(t_0)\right| U^\dagger(t,t_0).$$

The time-evolution operator must be unitary, i.e. $U^\dagger U = UU^\dagger = \hat{\mathbb{I}}$, so that the norm of a state remains constant over time, i.e.

$$\left\langle\Psi(t)\,\middle|\,\Psi(t)\right\rangle = \left\langle\Psi(t_0)\,\middle|\,U^\dagger(t,t_0)U(t,t_0)\,\middle|\,\Psi(t_0)\right\rangle = \left\langle\Psi(t_0)\,\middle|\,\Psi(t_0)\right\rangle.$$

Further, when $t = t_0$, the operator must be equal to the identity operator as

$$\left|\Psi(t_0)\right\rangle \overset{!}{=} U(t_0,t_0)\left|\Psi(t_0)\right\rangle.$$

As all operators are constant respect to time, the Hamiltonian in the SP is determined by the classical Hamiltonian H, thus the Schrödinger equation is

$$H\left|A(t);t_0\right\rangle_S = i\frac{\partial}{\partial t}\left|A(t);t_0\right\rangle_S,$$

where t_0 is the initial or boundary condition imposed on the system, which is often set to zero in quantum mechanics. Since the Hamiltonian is independent of time in the Schrödinger picture, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}H = 0$, then the Schrödinger equation has a formal solution of the form

$$\left|A(t);t_0\right\rangle_S = e^{-iH(t-t_0)}\left|A(t_0);t_0\right\rangle_S.$$

Proof 3.1: Solving the Schrödinger eqation

Letting $\left|A(t);t_0\right\rangle_S = \left|\Psi(t)\right\rangle_S$, the time-evolution equation is

$$\left|\Psi(t)\right\rangle = U(t)\left|\Psi(t_0)\right\rangle,$$

i.e. the final state $\left|\Psi(t)\right\rangle$ is the initial state $\left|\Psi(t_0)\right\rangle$ with some multiplicative factor $U(t)$. As such, the Schrödinger equation becomes

$$i\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle_S = H\left|\Psi(t)\right\rangle_S \to i\frac{\partial}{\partial t}U(t)\left|\Psi(t_0)\right\rangle_S = HU(t)\left|\Psi(t_0)\right\rangle_S.$$

Since the initial state $\left|\Psi(t_0)\right\rangle_S$ is a constant ket, this must obey

$$i\frac{\partial}{\partial t}U(t) = HU(t).$$

Solving this gives the factor as $U(t) = e^{-\int_{t_0}^{t} H \ \mathrm{d}t'} = e^{-iH(t-t_0)}$, hence

$$\begin{aligned}\left|\Psi(t)\right\rangle_S &= U(t)\left|\Psi(t_0)\right\rangle_S \\ \Rightarrow \left|A(t);t_0\right\rangle_S &= e^{-iH(t-t_0)}\left|A(t_0);t_0\right\rangle_S.\end{aligned}$$

For this picture, the Hamiltonians are denoted with a superscript S, i.e.

$$H \to H^S; \qquad H_0 \to H_0^S; \qquad H_{\text{int}} \to H_{\text{int}}^S.$$

3.1.2 The Heisenberg Picture

In contrast to the Schrödinger picture, in the Heisenberg picture (HP) states are time-independent while the operators are time-dependent. States in the Schrödinger and Heisenberg pictures must be equivalent, thus are related by

$$\left|A;t_0\right\rangle_H \overset{!}{=} \left|A(t_0);t_0\right\rangle_S = e^{-iH(t-t_0)}\left|A(t);t_0\right\rangle_S.$$

One must also ensure the invariance of elements between pictures, i.e.

$${}_H\left\langle B;t_0\right|\hat{O}^H(t)\left|A;t_0\right\rangle_H \overset{!}{=} {}_S\left\langle B(t);t_0\right|\hat{O}^S\left|A(t);t_0\right\rangle_S.$$

As such, the operators are related by

$$\hat{O}^H(t) = e^{iH(t-t_0)}\hat{O}^S e^{-iH(t-t_0)}.$$

Proof 3.2: Relating the Schrödinger and Heisenberg picture operators

The Schrödinger time-evolution equation for a ket $|A\rangle$ is given by

$$\left|A(t);t_0\right\rangle_S = e^{-iH(t-t_0)}\left|A(t_0);t_0\right\rangle_S \equiv e^{-iH(t-t_0)}\left|A;t_0\right\rangle_H.$$

Similarly, the equation for a bra $\langle B|$ is

$$ {}_S\langle B(t);t_0| = {}_S\langle B(t_0);t_0|e^{iH(t-t_0)} \equiv {}_H\langle B;t_0|e^{iH(t-t_0)}. $$

From the equivalency of the Heisenberg and Schrödinger pictures, then

$$ \begin{aligned} {}_S\Big\langle B(t);t_0 \Big| \hat{O}^S \Big| A(t);t_0 \Big\rangle_S &= {}_H\Big\langle B;t_0 \Big| e^{iH(t-t_0)}\hat{O}^S e^{-iH(t-t_0)} \Big| A;t_0 \Big\rangle_H \\ &\overset{!}{=} {}_H\Big\langle B;t_0 \Big| \hat{O}^H(t) \Big| A;t_0 \Big\rangle_H \\ \Rightarrow \hat{O}^H(t) &= e^{iH(t-t_0)}\hat{O}^S e^{-iH(t-t_0)}. \end{aligned} $$

For equivalency of the pictures, time-independent Hamiltonians of the systems must be equal, i.e. $H^H \overset{!}{=} H^S$. This can be seen by setting $\hat{O} \to H$ in the above such that the exponentials cancel. However, this is not generally true for the components of H, i.e. in general $H_0^H \neq H_0^S$ due to the commutation $\left[H_0^S, H_0^H\right] \neq 0$.

3.1.3 The Dirac Picture

The Dirac, or interaction, picture (IP) is an intermediate of the Heisenberg and Schrödinger pictures. Whereas in the other pictures either the state or operators carry time-dependence, in the interction picture both carry time-dependence. States in the interaction picture are defined in terms of those in the Schrödinger picture by

$$ \big|\Psi(t)\big\rangle_I \coloneqq U^\dagger(t)\big|\Psi(t)\big\rangle_S = e^{iH_0(t-t_0)}\big|\Psi(t)\big\rangle_S . $$

As a result, operators in the interaction picture are related to those in the Schrödinger picture by

$$ \hat{O}^I(t) = e^{iH_0^S(t-t_0)}\hat{O}^S e^{-iH_0^S(t-t_0)}. $$

Proof 3.3: Relating Schrödinger and interaction picture operators

If one were to set $H_{\text{int}} = 0$, one can see that the time-evolution of the exact part of the Hamiltonian H_0 is described by

$$i\,\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle = H_0(t)\left|\Psi(t)\right\rangle.$$

By setting $\left|\Psi(t)\right\rangle = U(t)\left|\Psi(t_0)\right\rangle$, where $\left|\Psi(t_0)\right\rangle$ is a constant ket, the above can be rearranged to

$$\frac{\partial}{\partial t}U(t) = -iH_0(t)U(t).$$

Most generally, the time-evolution operator is then given by

$$U(t) = e^{-i\int_{t_0}^{t} H_0(t')\ \mathrm{d}t'},$$

which for a time-independent H_0 is simply $U(t) = e^{-iH_0(t-t_0)}$. Imposing the invariance of matrix elements as in Proof 3.2, i.e.

$$\begin{aligned}{}_I\left\langle B(t);t_0\right|\hat{O}^I(t)\left|A(t);t_0\right\rangle_I &\overset{!}{=} {}_S\left\langle B(t);t_0\right|\hat{O}^S\left|A(t);t_0\right\rangle_S \\ &= {}_I\left\langle B(t);t_0\right|U^\dagger(t)\hat{O}^S U(t)\left|A(t);t_0\right\rangle_I,\end{aligned}$$

the interaction picture operators are then

$$\hat{O}^I(t) = U^\dagger(t)\hat{O}^S U(t) = e^{iH_0(t-t_0)}\hat{O}^S e^{-iH_0(t-t_0)}.$$

Using the relations

$$\begin{aligned}\left|A(t);t_0\right\rangle_S &= e^{-iH_0(t-t_0)}\left|A(t_0);t_0\right\rangle_S \\ e^{-iH(t-t_0)}\left|A;t_0\right\rangle_H &= \left|A(t);t_0\right\rangle_S = e^{-iH_0(t-t_0)}\left|A(t);t_0\right\rangle_I,\end{aligned}$$

one may derive further links between the pictures, e.g.

$$\begin{aligned}e^{-iH^S(t-t_0)}\left|A;t_0\right\rangle_H &= \left|A(t);t_0\right\rangle_S = e^{-iH_0^S(t-t_0)}\left|A(t);t_0\right\rangle_I \\ \Rightarrow \left|A(t);t_0\right\rangle_I &= \underbrace{e^{iH_0^S(t-t_0)}e^{-iH^S(t-t_0)}}_{\equiv U(t,t_0)}\left|A;t_0\right\rangle_H.\end{aligned}$$

Time-Evolution of Interaction Picture States

One may now ask how the Schrödinger equation evolves in time, i.e. $i\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle = \ ?$ One might be tempted to dive into calculations as follows:

$$\begin{aligned} i\,\frac{\partial}{\partial t}\left|A(t);t_0\right\rangle_I &= i\,\frac{\partial}{\partial t}\left(U(t,t_0)\left|A;t_0\right\rangle_H\right) \\ &= \left(-H_0^S U + H^S U\right)\left|A;t_0\right\rangle_H \\ &= \left(H^S - H_0^S\right)U\left|A;t_0\right\rangle_H \\ &= H_{\text{int}}^S\left|A(t);t_0\right\rangle_I, \end{aligned}$$

however, this is <u>incorrect</u> - one must be careful with how the Hamiltonian elements commute. The correct way to derive the Schrödinger equation for the interaction picture is as follows:

$$\begin{aligned} i\,\frac{\partial}{\partial t}\left|A(t);t_0\right\rangle_I &= i\,\frac{\partial}{\partial t}\left(e^{iH_0^S(t-t_0)}e^{-iH^S(t-t_0)}\left|A;t_0\right\rangle_H\right) \\ &= \left(e^{iH_0^S(t-t_0)}\underbrace{\left(-H_0^S+H^S\right)}_{=H_{\text{int}}^S}e^{-iH^S(t-t_0)}\right)\left|A;t_0\right\rangle_H \\ &= \left(e^{iH_0^S(t-t_0)}\left(H_{\text{int}}^S\right)e^{-iH_0^S(t-t_0)}\underbrace{e^{iH_0^S(t-t_0)}e^{-iH^S(t-t_0)}}_{\equiv U(t,t_0)}\right)\left|A;t_0\right\rangle_H \\ &= \underbrace{\left(e^{iH_0^S(t-t_0)}\left(H_{\text{int}}^S\right)e^{-iH_0^S(t-t_0)}\right)}_{:=H_{\text{int}}^I}\underbrace{U(t,t_0)\left|A;t_0\right\rangle_H}_{\equiv\left|A(t);t_0\right\rangle_I} \\ &= H_{\text{int}}^I\left|A(t);t_0\right\rangle_I. \end{aligned}$$

This is referred to as the Schwinger-Tomonaga equation. Separating the ket shown here into its constant initial ket and time-evolution operator, the above also gives

$$i\,\frac{\partial}{\partial t}U(t,t_0) = H_{\text{int}}^I U(t,t_0).$$

This may be solved iteratively using the boundary condition $\lim_{t \to t_0} U(t, t_0) \to 1$, i.e. $U(t_0, t_0) = e^0 e^0 = 1$.

Proof 3.4: Iteratively solving the Schrödinger equation

The Schrödinger equation can be rearranged as

$$\frac{\partial}{\partial t} U(t, t_0) = -i H^I_{\text{int}}(t) U(t, t_0).$$

Integrating both sides with respect to a dummy variable t' gives

$$\int_{t_0}^{t} \frac{\partial}{\partial t'} U(t', t'_0) \, \mathrm{d}t' = -i \int_{t_0}^{t} H^I_{\text{int}}(t') U(t', t'_0) \, \mathrm{d}t'.$$

Rewriting $U(t, t_0)$ as $U(t)$, then by carrying out the integration with the relation $U(t_0) = 1$, the time-evolution operation is hence

$$U(t) = 1 - i \int_{t_0}^{t} H^I_{\text{int}}(t') U(t') \, \mathrm{d}t'.$$

The time-evolution operator in the integral on the right-hand side can be substituted by the same equation under a new dummy variable, i.e.

$$\begin{aligned} U(t) &= 1 - i \int_{t_0}^{t} H^I_{\text{int}}(t') U(t') \, \mathrm{d}t' \\ &= 1 - i \int_{t_0}^{t} H^I_{\text{int}}(t') \left(1 - i \int_{t_0}^{t_1} H^I_{\text{int}}(t'') U(t'') \, \mathrm{d}t'' \right) \, \mathrm{d}t' \\ &= 1 - i \int_{t_0}^{t} H^I_{\text{int}}(t') \, \mathrm{d}t' + (-i)^2 \int_{t_0}^{t} \int_{t_0}^{t_1} H^I_{\text{int}}(t') H^I_{\text{int}}(t'') U(t'') \, \mathrm{d}t'' \, \mathrm{d}t'. \end{aligned}$$

This process may be continued indefinitely by replacing the resulting time-evolution operator within the integral. The result of each additional iteration is hence a value that is closer converged to the true value.

The Time-Ordered Product (TOP)

The iterative solution can be written more compactly were it not for the fact that the integral bounds are different and that the ordering matters. This is due to the fact that the upper bounds must be equal, i.e. all iterations of the solution are considered over the same time period. By introducing the time-ordered product (TOP), one can ensure that $H^I_{\text{int}}(t)$ acts as the proper time and hence preserves causality. The TOP, denoted T, for two operators $\phi(t_1)$ and $\phi(t_2)$ is defined as

$$T\left\{\phi(t_1), \phi(t_2)\right\} = \begin{cases} \phi(t_1)\phi(t_2) & t_1 > t_2 \\ \phi(t_2)\phi(t_1) & t_2 > t_1. \end{cases}$$

For n operators there are $n!$ permutations which must be considered, only one of which obeys causality. In such a case, all terms are scaled by a factor of $\frac{1}{n!}$ in order to normalise the result. The TOP may alternatively be written in terms of a Heaviside step function Θ, which is defined by

$$\Theta(a-b) \coloneqq \begin{cases} 1 & a > b \\ \frac{1}{2} & a = b \\ 0 & a < b. \end{cases}$$

The TOP for two operators is then

$$T\left\{\phi(t_1), \phi(t_2)\right\} = \Theta(t_1 - t_2)\phi(t_1)\phi(t_2) + \Theta(t_2 - t_1)\phi(t_2)\phi(t_1).$$

This allows the iterative solution to be written

$$U(t, t_0) = T\left\{\exp\left(-i\int_{t_0}^{t} H^I_{\text{int}}(t')\ \mathrm{d}t'\right)\right\}.$$

Proof 3.5: Time-evolution operator in terms of the time-ordered product

It has already been shown that the Schrödinger equation for the time-evolution operator in the interaction picture is

$$\frac{\partial}{\partial t} U(t, t_0) = -iH^I_{\text{int}} U(t, t_0),$$

which one can immediately see has an exponential solution. Solving this iteratively then gives

$$\begin{aligned}
U(t, t_0) &= 1 - i \int_{t_0}^{t} H^I_{\text{int}}(t') U(t', t'_0)\ \mathrm{d}t' \\
&= 1 - i \int_{t_0}^{t} H^I_{\text{int}}(t') \left(1 - i \int_{t_0}^{t} H^I_{\text{int}}(t'') U(t'', t''_0)\ \mathrm{d}t'' \right)\ \mathrm{d}t' \\
&= 1 - i \int_{t_0}^{t} H^I_{\text{int}}\left(t'\right)\ \mathrm{d}t' \\
&\qquad + \frac{(-i)^2}{2!} \int_{t_0}^{t} \int_{t_0}^{t_1} H^I_{\text{int}}\left(t'\right) H^I_{\text{int}}\left(t''\right) U(t'', t''_0)\ \mathrm{d}t'\ \mathrm{d}t'' \\
&\qquad\qquad \vdots \\
&= \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} \mathrm{d}t' \ldots \int_{t_0}^{t_{n-1}} \mathrm{d}t^{(n)} \left(H^I_{\text{int}}\left(t'\right) \ldots H^I_{\text{int}}\left(t^{(n)}\right) \right) \\
&= T \left\{ \exp\left(-i \int_{t_0}^{t} H^I_{\text{int}}\left(t'\right)\ \mathrm{d}t' \right) \right\}
\end{aligned}$$

When using the TOP, the Hamiltonains H^I are also time-ordered. Furthermore, all integral bounds must be equal, otherwise[...]. When imposing the same boundary values in two dimensions, the area doubles (= 2!), whereas in three dimensions the area increases six-fold (= 3!). Therefore, by introducing the TOP, it allows for all upper bounds to be equal. However, the TOP only works if the Hamiltonian contains an even number of fermionic factor, as in QED, otherwise the reordering process introduces a factor of -1.

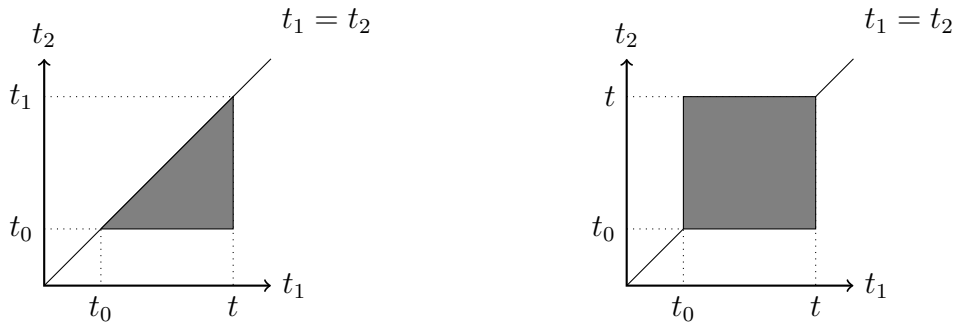


Figure 3.1: Left: Different boundaries. Area shaded is covered by the integral $\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2$. Right: Same boundaries. Area shaded is covered by the integral $\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t} \mathrm{d}t_2$.

The S-Matrix

In the interaction picture, the transition amplitude is given by

$$ {}_I\left\langle f; t_f \,\middle|\, U\left(t_f, t_i\right) \,\middle|\, i; t_i \right\rangle_I . $$

For a scattering process in which an initial state i becomes a final state f, one is interested in how the system evolves over long time periods, i.e. when $t_{i/f} \to \mp\infty$. In this case, the relevant unitary evolution operator is the **S-matrix operator**, which is defined by

$$ S_{fi} \coloneqq \lim_{t_{i/f} \to \mp\infty} U(t_f, t_i) = U(\infty, -\infty). $$

Writing the interaction Hamiltonian H_{int}^I as an integral over all space of the Hamiltonain density $\mathscr{H}_{\mathrm{int}}^I$, i.e. $H_{\mathrm{int}}^I = \int_{-\infty}^{\infty} \mathrm{d}^3\mathbf{x}' \; \mathscr{H}_{\mathrm{int}}^I\left(\mathbf{x}'\right)$, the S-matrix can then be written as

$$ S = T\left\{ \exp\left(-i \int_{-\infty}^{\infty} \mathrm{d}^4 x' \; \mathscr{H}_{\mathrm{int}}^I(x') \right) \right\}, $$

where x' is the four-position over which the Hamiltonian density is integrated. Using the adiabatic approximation $\lim_{t \to \mp\infty} H_{\mathrm{int}}(t) = 0$, the interaction picture

can be written in terms of the Heisenberg picture as

$$\begin{aligned}|\mathbf{p};\mp\infty\rangle_I &= z^{\frac{1}{2}}\,|\mathbf{p};\mp\infty\rangle_H \\ &= z^{\frac{1}{2}}\,|\mathbf{p}\rangle_H\,,\end{aligned}$$

where $z^{\frac{1}{2}}$ is the wavefunction normalisation of the asymptotic state $|\mathbf{p}\rangle$. This result is useful as states in the Heisenberg picture are manifestly stationary, and so it is now possible to ignore the trivial solution $|i\rangle = |f\rangle = |0\rangle$.

3.2 The Feynman Propagator

The time-ordered product is a mathematical representation of causality, i.e. a particle must be created before it can be destroyed. The calculation of a time-ordered product in Minkowski space is

$$T\left\{\phi(x),\phi(y)\right\} = \Theta\left(x^0-y^0\right)\phi(x)\phi(y) + \Theta\left(y^0-x^0\right)\phi(y)\phi(x),$$

where the Heaviside step function Θ is defined by

$$\Theta(a-b) \coloneqq \begin{cases} 1 & a > b \\ \frac{1}{2} & a = b \\ 0 & a < b. \end{cases}$$

One should first observe that each particle has forward and backward components that do not commute (due to non-commutativity of creation and annihilation operators), i.e.

$$\begin{gathered}\phi(x) = \underbrace{\phi^+(x)}_{\propto a_p^\dagger e^{ip\cdot x}} + \underbrace{\phi^-(x)}_{\propto a_p e^{-ip\cdot x}}\,; \\ \left[\phi^+(x),\phi^+(y)\right] = \left[\phi^-(x),\phi^-(y)\right] = 0.\end{gathered}$$

It can hence be seen that the commutation between two particles is the difference of two 4D Pauli-Jordan operators, i.e.

$$\left[\phi(x),\phi(y)\right] = -\underbrace{\Delta^-(x-y)}_{\substack{\text{retarded}\\\text{propagator}}} + \underbrace{\Delta^+(x-y)}_{\substack{\text{advanced}\\\text{propagator}}}.$$

Proof 3.6: Commutation of scalar fields

$$
\begin{aligned}
\left[\phi(x),\phi(y)\right] &= \left[\phi^+(x)+\phi^-(x),\phi^+(y)+\phi^-(y)\right] \\
&= \left[\phi^+(x),\phi^+(y)\right]+\left[\phi^+(x),\phi^-(y)\right] \\
&\qquad +\left[\phi^-(x),\phi^+(y)\right]+\left[\phi^-(x),\phi^-(y)\right] \\
&\propto \underbrace{\left[a^\dagger_{\mathbf{p}},a^\dagger_{\mathbf{p}}\right]}_{=0}+\left[a^\dagger_{\mathbf{p}},a_{\mathbf{p}}\right]+\left[a_{\mathbf{p}},a^\dagger_{\mathbf{p}}\right]+\underbrace{\left[a_{\mathbf{p}},a_{\mathbf{p}}\right]}_{=0} \\
&= \left[\phi^+(x),\phi^-(y)\right]+\left[\phi^-(x),\phi^+(y)\right] \\
&= \int_{\mathbf{p}}\left[a^\dagger e^{ip\cdot x},ae^{-ip\cdot y}\right]+\int_{\mathbf{p}}\left[ae^{-ip\cdot x},a^\dagger e^{ip\cdot y}\right] \\
&= \int_{\mathbf{p}}\underbrace{\left[a^\dagger,a\right]}_{=-1}e^{ip\cdot(x-y)}+\int_{\mathbf{p}}\underbrace{\left[a,a^\dagger\right]}_{=1}e^{-ip\cdot(x-y)} \\
&= -\underbrace{\int_{\mathbf{p}}e^{ip\cdot(x-y)}}_{\equiv\Delta^-(x-y)}+\underbrace{\int_{\mathbf{p}}e^{-ip\cdot(x-y)}}_{\equiv\Delta^+(x-y)} \\
&= -\Delta^-(x-y)+\Delta^+(x-y).
\end{aligned}
$$

Some new shorthand will now be introduced:

$$
\begin{aligned}
\phi_x \equiv \phi(x) &= \phi^+_x+\phi^-_x \\
\left.\begin{aligned}\Theta_> &\equiv \Theta(x^0-y^0)\\ \Theta_< &\equiv \Theta(y^0-x^0)\end{aligned}\right\} &\Rightarrow \Theta_>+\Theta_<=1
\end{aligned}
$$

The TOP of two scalar fields is then given by

$$
T\left\{\phi_x,\phi_y\right\} =:\phi_x\phi_y: +\Theta_>\Delta^+(x-y)-\Theta_<\Delta^-(x-y).
$$

Proof 3.7: TOP of two scalar fields

$$\begin{aligned} T\left\{\phi_x, \phi_y\right\} &= \Theta_> \left(\phi_x^+ + \phi_x^-\right)\left(\phi_y^+ + \phi_y^-\right) + \Theta_> \left(\phi_y^+ + \phi_y^-\right)\left(\phi_x^+ + \phi_x^-\right) \\ &= \Theta_> \left(\phi_x^+\phi_y^+ + \phi_x^-\phi_y^-\right) + \Theta_< \left(\phi_y^+\phi_x^+ + \phi_y^-\phi_x^-\right) \\ &\quad + \Theta_> \left(\phi_x^-\phi_y^+ + \phi_x^+\phi_y^-\right) + \Theta_< \left(\phi_y^-\phi_x^+ + \phi_y^+\phi_x^-\right). \end{aligned}$$

Writing $\phi_x^-\phi_y^+ = \left[\phi_x^-, \phi_y^+\right] + \phi_y^+\phi_x^-$ and $\phi_y^-\phi_x^+ = \left[\phi_y^-, \phi_x^+\right] + \phi_x^+\phi_y^-$, the TOP is then given by

$$\begin{aligned} T\left\{\phi_x, \phi_y\right\} &= \underbrace{(\Theta_> + \Theta_<)}_{=1} \underbrace{\left(\phi_x^+\phi_y^+ + \phi_x^+\phi_y^- + \phi_x^-\phi_y^+ + \phi_x^-\phi_y^-\right)}_{=:\phi_x\phi_y:} \\ &\quad + \Theta_> \underbrace{\left[\phi_x^-, \phi_y^+\right]}_{=\Delta^+(x-y)} - \Theta_< \underbrace{\left[\phi_x^+, \phi_y^-\right]}_{=\Delta^-(x-y)} \\ &=:\phi_x\phi_y: + \Theta_>\Delta^+(x-y) - \Theta_<\Delta^-(x-y). \end{aligned}$$

Since $\left\langle 0 \middle| :\phi_x\phi_y: \middle| 0 \right\rangle = 0$, the vacuum expectation value (VEV) of the TOP is then

$$\begin{aligned} \left\langle 0 \middle| T\left\{\phi_x, \phi_y\right\} \middle| 0 \right\rangle &= \left\langle 0 \middle| :\phi_x\phi_y: + \Theta_>\Delta^+(x-y) - \Theta_<\Delta^-(x-y) \middle| 0 \right\rangle \\ &= \left\langle 0 \middle| :\phi_x\phi_y: \middle| 0 \right\rangle + \left\langle 0 \middle| \Theta_>\Delta^+(x-y) \middle| 0 \right\rangle \\ &\qquad - \left\langle 0 \middle| \Theta_<\Delta^-(x-y) \middle| 0 \right\rangle \\ &= \Theta_>\Delta^+(x-y)\left\langle 0 \middle| 0 \right\rangle - \Theta_<\Delta^-(x-y)\left\langle 0 \middle| 0 \right\rangle \\ &= \Theta_>\Delta^+(x-y) - \Theta_<\Delta^-(x-y), \end{aligned}$$

which is the definition of the Feynman propagator $i\Delta_F(x-y)$, i.e.

$$i\Delta_F(x-y) \coloneqq \Theta_>\Delta^+(x-y) - \Theta_<\Delta^-(x-y).$$

Graphically, the Feynman propagator may be represented by a scalar field propagating from an initial point x to a final point y, as shown in Figure 3.2.

Figure 3.2: Graphical representation of the Feynman propagator for a scalar field with momentum k.

3.2.1 Calculating $i\Delta_F(x-y)$

The Feynman propagator $i\Delta_F(x-y)$ has now been introduced, however it still not at a point whereby it can be used. The propagator may, however, be quantified for use in calculations.

Starting from the definition of the Feynman propagator, expand the propgators and use the fact that the zeroth component of four-momenta is energy, i.e. $k^0 = E_k$;

$$\begin{aligned} i\Delta_F(x-y) &= \Theta_> \Delta^+(x-y) - \Theta_< \Delta^-(x-y) \\ &= \Theta_> \Delta^+(x-y) + \Theta_< \Delta^+(y-x) \\ &= \int_{\mathbf{k}} \Theta_> e^{-ik\cdot(x-y)} + \Theta_< e^{-ik\cdot(y-x)} \\ &= \int_{\mathbf{k}} \underbrace{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}_{\text{spatial}} \underbrace{\left(\Theta_> e^{-iE_k\left(x^0-y^0\right)} + \Theta_< e^{-iE_k\left(y^0-x^0\right)}\right)}_{\text{temporal}}. \end{aligned}$$

Recall Cauchy's residue theorem for a line integral in the z plane over a curve C with a pole located at a:

$$\begin{aligned} &\text{i)} \quad \textstyle\int_C f(z)\ \mathrm{d}z = 0 \\ &\text{ii)} \quad \textstyle\oint_C \frac{f(z)}{z-a}\ \mathrm{d}z = 2\pi i f(a) \\ &\qquad \Rightarrow f(a) = \textstyle\oint_C \frac{\mathrm{d}z}{2\pi i}\frac{f(z)}{z-a}. \end{aligned}$$

Consider the temporal pole, i.e.

$$f(a) = e^{-iE_k\left(x^0-y^0\right)} = \oint \frac{\mathrm{d}k^0}{2\pi i} \frac{e^{-ik^0\left(x^0-y^0\right)}}{\left(k^0\right)^2 - \left(E_k - i\epsilon\right)^2},$$

then one may note that $\left(k^0\right)^2 - \left(E_k - i\epsilon\right)^2 = \left(k^0 - \left(E_k - i\epsilon\right)\right)\left(k^0 + \left(E_k - i\epsilon\right)\right)$, and hence can see that two poles exist; one at $k^0 = E_k - i\epsilon$ and the other at

$k^0 = -(E_k - i\epsilon)$. One can approximate the pole using $\lim_{\epsilon\to 0}\epsilon^2 = 0$, i.e.

$$(E_k - i\epsilon)^2 = E_k^2 - 2iE_k\epsilon - \epsilon^2 \approx E_k^2 - 2iE_k\epsilon.$$

Defining $\epsilon' \coloneqq 2E_k\epsilon$, then $(E_k - i\epsilon)^2 \approx E_k^2 - i\epsilon'$, which covers all of space $[-\infty, \infty]$ in the energy/temporal dimension. The Feynman propagator is hence

$$\begin{aligned} i\Delta_F(x-y) &= \int_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-ik^0(x^0-y^0)} \frac{i}{k^2 - m^2 + i\epsilon'} \,\mathrm{d}k^0 \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}k^4}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2 - m^2 + i\epsilon'}. \end{aligned}$$

3.3 Wick's Theorem

Wick's theorem is used in quantum field theory to reduce arbitrary numbers of creation and annihilation operators to sums of products of pairs of these operators. The theorem states

$$\begin{aligned} T\left\{\phi(x_1)\phi(x_2)\dots\phi(x_n)\right\} &\equiv T\left\{\phi_1\phi_2\dots\phi_n\right\} \\ &=: \{\phi_1\phi_2\dots\phi_n + \text{all possible contractions}\}: . \end{aligned}$$

The contraction of two operators, denoted by *bullets* $\bullet$, is defined by

$$\hat{A}^{\bullet}\hat{B}^{\bullet} \coloneqq \hat{A}\hat{B} - :\hat{A}\hat{B}: .$$

Proof 3.7 was Wick's theorem for $n = 2$, which will now be extended to the $n = 4$ case.

$$\begin{aligned} T\left\{\phi_1\phi_2\phi_3\phi_4\right\} =: \phi_1\phi_2\phi_3\phi_4 &+ \phi_1^{\bullet}\phi_2^{\bullet}\phi_3^{\bullet\bullet}\phi_4^{\bullet\bullet} + \phi_1^{\bullet}\phi_2^{\bullet\bullet}\phi_3^{\bullet}\phi_4^{\bullet\bullet} + \phi_1^{\bullet}\phi_2^{\bullet\bullet}\phi_3^{\bullet\bullet}\phi_4^{\bullet} \\ &+ \phi_1^{\bullet}\phi_2^{\bullet}\phi_3\phi_4 + \phi_1^{\bullet}\phi_2\phi_3^{\bullet}\phi_4 + \phi_1^{\bullet}\phi_2\phi_3\phi_4^{\bullet} + \phi_1\phi_2^{\bullet}\phi_3^{\bullet}\phi_4 + \phi_1\phi_2^{\bullet}\phi_3\phi_4^{\bullet} + \phi_1\phi_2\phi_3^{\bullet}\phi_4^{\bullet} : . \end{aligned}$$

As the vacuum expectation value of any normal-ordered operators vanish, i.e. $\left\langle 0 \middle| :\hat{O}: \middle| 0 \right\rangle$, inserting any term in which there remains uncontracted

field operators gives zero. This means that only the three fully contracted terms survive:

$$\begin{aligned}\left\langle 0 \left| T\left\{\phi_1\phi_2\phi_3\phi_4\right\} \right| 0 \right\rangle &= \langle 0 \,|\, \phi_1^{\bullet}\phi_2^{\bullet}\phi_3^{\bullet\bullet}\phi_4^{\bullet\bullet} + \phi_1^{\bullet}\phi_2^{\bullet\bullet}\phi_3^{\bullet}\phi_4^{\bullet\bullet} + \phi_1^{\bullet}\phi_2^{\bullet\bullet}\phi_3^{\bullet\bullet}\phi_4^{\bullet} \,|\, 0\rangle \\ &= D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23},\end{aligned}$$

where

$$\phi_i^{\bullet}\phi_j^{\bullet} = \left\{ \begin{matrix} \left[\phi_i^+, \phi_j^-\right] & i^0 > j^0 \\ \left[\phi_j^+, \phi_i^-\right] & i^0 < j^0 \end{matrix} \right\} = D_{ij},$$

where $D_{ij} = i\Delta_F\left(x_i - x_j\right)$ is the Feynman propagator for two scalar fields.

At this point Wick's theorem has been derived for the $n = 2$ and $n = 4$ cases. This can now be extended to a general number n of field operators. Firstly, assume that Wick's theorem holds for $n-1$ and attempt to show that it holds for n operators. Without loss of generality, one can assume that $x_1^0 > x_2^0 > \ldots x_n^0$ because if this is not true one can relabel the temporal components in such a way that both sides remain unchanged. Then, consider the TOP of n operators, assuming the latter $n-1$ operators hold Wick's theorem, i.e.

$$\begin{aligned}T\left\{\phi_1\phi_2\ldots\phi_n\right\} &= \phi_1 :\phi_2\ldots\phi_n + \text{contractions w/o } \phi_1: \\ &= \left(\phi_1^+\phi_1^-\right) :\phi_2\ldots\phi_n + \text{contractions w/o } \phi_1: .\end{aligned}$$

At this point, we wish to move $\phi_1^{\pm}$ into the normal ordering with the rest of the fields, in such a way that it still holds. The ϕ_1^+ term is more complicated as one must bring it into the normal ordering, requiring that ϕ_1^+ commutes with all terms that are already normally ordered, i.e. consider the terms without contractions

$$\begin{aligned}\phi_1^+ :\phi_2\ldots\phi_n: &= :\phi_2\ldots\phi_n: + \left[\phi_1^+, :\phi_2, \ldots\phi_n:\right] \\ &= :\phi_2\ldots\phi_n: + : \Big(\left[\phi_1^+, \phi_2\right]\phi_3\ldots\phi_n \\ &\qquad + \phi_2\left[\phi_1^+, \phi_3\right]\phi_4\ldots\phi_n \\ &\qquad\qquad \vdots \\ &\qquad + \phi_2\phi_3\ldots\left[\phi_1^+, \phi_n\right] : \\ &= :\big(\phi_1^+\phi_2\ldots\phi_n + \phi_1^{+\bullet}\phi_2^{\bullet}\ldots\phi_n + \ldots + \phi_1^{+\bullet}\phi_2\ldots\phi_n^{\bullet}\big): .\end{aligned}$$

The first term combines with the ϕ_1^- term such that

$$\phi_1^- :(\phi_2 \dots \phi_n):=:\phi^- \phi_2 \dots \phi_n:$$

and hence the overall normal ordering is now

$$\begin{aligned}\left(\phi_1^+ + \phi_1^-\right) :(\phi_2 \dots \phi_n): &=:\left(\phi_1^+ + \phi_1^-\right)(\phi_2 \dots \phi_n) + \text{other terms}: \\ &=:\phi_1 \phi_2 \dots \phi_n + \text{other terms}: .\end{aligned}$$

By repeating this process for $\phi_2 \dots \phi_n$, one obtains all possible contractions of all fields, including those of ϕ_1.

3.4 Transition Amplitudes and Feynman Rules

3.4.1 Transition Amplitude for $\phi(q) \to \chi^+(k)\chi^-(p)$

Consider the toy model of a real scalar field ϕ with momentum q coupling to two complex scalar fields $\chi^\pm$ with respective momenta k and p. The Lagrangian density $\mathcal{L}$ for such a system is

$$\mathcal{L} = \underbrace{\frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2} m_\phi^2 \phi^2}_{\substack{\text{Klein-Gordon of real} \\ \text{scalar field } \phi}} + \underbrace{\frac{1}{2}\left(\partial_\mu \chi\right)\left(\partial^\mu \chi\right) - m_\chi^2 \chi^\dagger \chi}_{\substack{\text{Klein-Gordon of complex} \\ \text{scalar fields } \chi^\pm}} - \underbrace{g \chi^\dagger \chi \phi}_{\mathscr{H}_{\text{int}}}.$$

The S-matrix amplitude is given by

$$\begin{aligned} S &= T\left\{\exp\left[-i \int \mathrm{d}^4 x \; \mathscr{H}_{\text{int}}\right]\right\} \\ &= \mathbb{I} - i \int \mathscr{H}_{\text{int}} \; \mathrm{d}^4 x + \frac{1}{2}(-i)^2 \iint \mathscr{H}_{\text{int}_1} \mathscr{H}_{\text{int}_2} \; \mathrm{d}^4 x_2 \; \mathrm{d}^4 x_1 \\ &= \underbrace{\mathbb{I}}_{\substack{\text{does not} \\ \text{contribute}}} - i \int g \chi^\dagger \chi \phi \; \mathrm{d}^4 x + \underbrace{\frac{1}{2}(-i)^2 \iint \left(g_1 \chi_1^\dagger \chi_1 \phi_1\right)\left(g_2 \chi_2^\dagger \chi_2 \phi_2\right) \; \mathrm{d}^4 x_2 \; \mathrm{d}^4 x_1}_{\int \mathcal{O}(g^2) \; \mathrm{d}^4 x \;\Rightarrow\; \text{small contribution}} \\ &\approx -i \int g \chi^\dagger \chi \phi \; \mathrm{d}^4 x + \mathcal{O}(g^2). \end{aligned}$$

The S-matrix element is hence

$$\begin{aligned}
S_{fi} &= \langle f \,|\, S \,|\, i \rangle \\
&= \left\langle \chi^+(p)\chi^-(k) \middle| T\left\{ \exp\left[-i \int \mathscr{H}_{\text{int}} \, \mathrm{d}^4x \right] \right\} \middle| \phi(q) \right\rangle \\
&= \left\langle 0 \middle| a_\chi(p) b_\chi(k) T\left\{ \exp\left[-i \int \mathscr{H}_{\text{int}} \, \mathrm{d}^4x \right] \right\} a^\dagger_\phi(q) \middle| 0 \right\rangle \\
&\approx -ig \int \left\langle 0 \middle| a^\bullet_\chi(p) b^{\bullet\bullet}_\chi(k) \chi^{\dagger\bullet}(x) \chi^{\bullet\bullet}(x) \phi^{\bullet\bullet\bullet}(x) a^{\dagger\bullet\bullet\bullet}_\phi(q) \middle| 0 \right\rangle + \mathcal{O}\left(g^2\right) \, \mathrm{d}^4x.
\end{aligned}$$

One is only interested in non-vanishing commutators resulting from Wick contractions. As such, consider the contracted pairs within the vacuum, e.g.

$$\begin{aligned}
\left\langle 0 \middle| \phi(x) a^\dagger_\phi(q) \middle| 0 \right\rangle &= \left\langle 0 \middle| \left(\int_{\mathbf{k}} a_\phi(k) e^{-ikx} \right) a^\dagger_\phi(q) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \int_{\mathbf{k}} e^{-ikx} \left(a_\phi(k) a^\dagger_\phi(q) \right) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \int_{\mathbf{k}} e^{-ikx} \left(\underbrace{a^\dagger_\phi(q) a_\phi(k)}_{\substack{\text{annihilates} \\ \text{vacuum}}} + \underbrace{\left[a_\phi(k), a^\dagger_\phi(q) \right]}_{= (2\pi)^3 2E_k \delta^{(3)}(\mathbf{k}-\mathbf{q})} \right) \middle| 0 \right\rangle \\
&= e^{-iqx}.
\end{aligned}$$

Similarly;

$$\begin{aligned}
\left\langle 0 \middle| a_\chi(p) \chi^+(x) \middle| 0 \right\rangle &= e^{ipx} \\
\left\langle 0 \middle| b_\chi(k) \chi^-(x) \middle| 0 \right\rangle &= e^{ikx}.
\end{aligned}$$

Overall, the matrix element is then

$$\begin{aligned}
S_{fi} &= -ig \left\langle 0 \middle| \int e^{ipx} e^{ikx} e^{-iqx} \, \mathrm{d}^4x \middle| 0 \right\rangle \\
&= -ig \left\langle 0 \middle| \int e^{i(p+k-q)x} \, \mathrm{d}^4x \middle| 0 \right\rangle \\
&= (-ig) \left(2\pi\right)^4 \delta^{(4)}\left(p + k - q\right) + \mathcal{O}(g^2).
\end{aligned}$$

3.4.2 Transition Amplitude for $\chi^+(p_1)\chi^-(p_2) \to \chi^+(k_1)\chi^-(k_2)$

It has already been shown that the S-matrix amplitude for a single real scalar field ϕ coupling to two complex scalar fields $\chi^\pm$ is given by

$$S = \mathbb{I} - i \int g\chi^\dagger \chi \phi \, \mathrm{d}^4x + \mathcal{O}(g^2).$$

However, this is specific and one wishes for a generalised form. Define the transition amplitude T_{fi}, usually denoted iT_{fi}, such that

$$\begin{aligned} S_{fi} &= \mathbb{I} + iT_{fi} \\ \Rightarrow iT_{fi} &\coloneqq (S - \mathbb{I})_{fi} \, . \end{aligned}$$

Now consider the scattering process $\chi^+(p_1)\chi^-(p_2) \to \chi^+(k_1)\chi^-(k_2)$, as shown in Figure 3.3. The transition amplitude of this is then given by

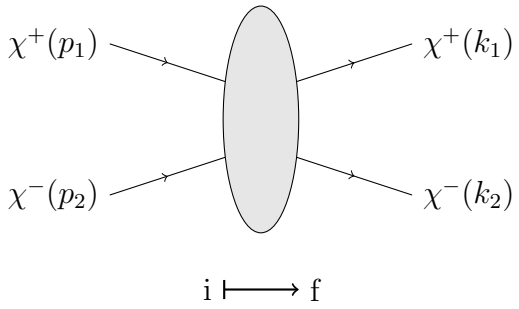


Figure 3.3: Scattering process of two complex scalar fields.

$$\begin{aligned} iT_{fi} &= (S - \mathbb{I})_{fi} \\ &= \langle f \,|\, S - \mathbb{I} \,|\, i \rangle \\ &= \left\langle k_1, k_2 \left| \, e^{-i \int \mathscr{H}_{\text{int}} \, \mathrm{d}^4x} \, \right| p_1, p_2 \right\rangle . \end{aligned}$$

In this instance the initial state is two fields $\chi\chi$ instead of ϕ, the creation and annihilation operators commute in expanding S, thus the first order

expansion in g is zero. One must then expand S to second order, i.e.

$$\begin{aligned} S &= T\left\{\exp\left[-i\int \mathscr{H}_{\text{int}}\ \mathrm{d}^4x\right]\right\} \\ &= \mathbb{I} + \underbrace{\mathcal{O}(g)}_{=0} + \frac{(-ig)^2}{2!}\iint \mathscr{H}_{\text{int}}(x)\mathscr{H}_{\text{int}}(y)\ \mathrm{d}^4y\ \mathrm{d}^4x + \mathcal{O}(g^3) \\ \Rightarrow S - \mathbb{I} &\approx \frac{(-ig)^2}{2!}\iint \mathscr{H}_{\text{int}}(x)\mathscr{H}_{\text{int}}(y)\ \mathrm{d}^4y\ \mathrm{d}^4x. \end{aligned}$$

The transition amplitude is then

$$\begin{aligned} iT_{fi} &= \langle k_1, k_2 \,|\, S - \mathbb{I} \,|\, p_1, p_2\rangle \\ &= \frac{(-ig)^2}{2!}\left\langle 0\left|\, a_{k_1}b_{k_2}\iint T\left\{\chi_x^\dagger\chi_x\phi_x\chi_y^\dagger\chi_y\phi_y\right\}\ \mathrm{d}^4y\ \mathrm{d}^4x\ a_{p_1}^\dagger b_{p_2}^\dagger\,\right|0\right\rangle. \end{aligned}$$

As $T\left\{\chi_x\chi_y\right\} = :\phi_x\phi_y: \Rightarrow \left\langle 0\left|\, T\left\{\chi_x\chi_y\right\}\,\right|0\right\rangle = 0$, the only non-zero contraction is $\phi_x^\bullet\phi_y^\bullet$, thus the ϕ-propagator is

$$\left\langle 0\left|\, T\left\{\phi_x\phi_y\right\}\,\right|0\right\rangle = i\Delta_F(x-y).$$

3.4.3 Wick Contractions

Knowing that the contraction under consideration is $\phi_x^\bullet\phi_y^\bullet$ allows those terms to be ignored until the vacuum has been removed, at which point a factor of $i\Delta_F(x-y)$ is introduced. In momentum space, this is given by

$$i\Delta_F(x-y) = \int \frac{\mathrm{d}^4q}{(2\pi)^4}e^{-iq\cdot(x-y)}\frac{i}{q^2 - m^2 + i\epsilon},$$

as earlier derived, where m is the mass of the intermediary particle ϕ from the Lagrangian density. In order to bring the creation and annihilation operators $a_{k_1,p_1}^{(\dagger)}$ and $b_{k_2,p_2}^{(\dagger)}$ into normal ordering, consider all possible contraction

permutations:

$$
\begin{aligned}
\left\langle 0 \middle| a_{k_1} b_{k_2} \chi_x^\dagger \chi_x \chi_y^\dagger \chi_y a_{p_1}^\dagger b_{p_2}^\dagger \middle| 0 \right\rangle &= \left\langle 0 \middle| a_{k_1} b_{k_2} \chi_x^\dagger \chi_x \chi_y^\dagger \chi_y a_{p_1}^\dagger b_{p_2}^\dagger \middle| 0 \right\rangle \\
&\quad + \left\langle 0 \middle| a_{k_1} b_{k_2} \chi_x^\dagger \chi_x \chi_y^\dagger \chi_y a_{p_1}^\dagger b_{p_2}^\dagger \middle| 0 \right\rangle \\
&\quad + \left\langle 0 \middle| a_{k_1} b_{k_2} \chi_x^\dagger \chi_x \chi_y^\dagger \chi_y a_{p_1}^\dagger b_{p_2}^\dagger \middle| 0 \right\rangle \\
&\quad + \left\langle 0 \middle| a_{k_1} b_{k_2} \chi_x^\dagger \chi_x \chi_y^\dagger \chi_y a_{p_1}^\dagger b_{p_2}^\dagger \middle| 0 \right\rangle \\
&= e^{ik_1x} e^{ik_2x} e^{-ip_1y} e^{-ip_2y} \\
&\quad + e^{ik_1x} e^{ik_2y} e^{-ip_1x} e^{-ip_2y} \\
&\quad + e^{ik_1y} e^{ik_2x} e^{-ip_1y} e^{-ip_2x} \\
&\quad + e^{ik_1y} e^{ik_2y} e^{-ip_1x} e^{-ip_2x} \\
&= e^{i(k_1+k_2)x} e^{-i(p_1+p_2)y} \\
&\quad + e^{i(k_1-p_1)x} e^{i(k_2-p_2)y} \\
&\quad + e^{i(k_2-p_2)x} e^{i(k_1-p_1)y} \\
&\quad + e^{-i(p_1+p_2)x} e^{i(k_1+k_2)y}.
\end{aligned}
$$

The transition amplitude is then

$$\begin{aligned}
iT_{fi} &= \frac{(-ig)^2}{2!}\left\langle 0\left|\, a_{k_1}b_{k_2}\iint T\left\{\chi_x^\dagger\chi_x\chi_y^\dagger\chi_y\right\}\,\mathrm{d}^4y\,\mathrm{d}^4x\; a_{p_1}^\dagger b_{p_2}^\dagger\,\right|0\right\rangle \\
&= \frac{(-ig)^2}{2!}\iint i\Delta_F(x-y)\left(e^{i(k_1+k_2)x}e^{-i(p_1+p_2)y}+e^{i(k_1-p_1)x}e^{i(k_2-p_2)y}\right. \\
&\qquad\qquad \left. +\, e^{i(k_2-p_2)x}e^{i(k_1-p_1)y}+e^{-i(p_1+p_2)x}e^{i(k_1+k_2)y}\right)\,\mathrm{d}^4y\,\mathrm{d}^4x \\
&= \frac{(-ig)^2}{2!}\iiint \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{i}{q^2-m^2+i\epsilon}e^{-iq\cdot(x-y)}\,(\text{exponential terms})\,\mathrm{d}^4y\,\mathrm{d}^4x \\
&= \frac{(-ig)^2}{2!}(2\pi)^4\int \mathrm{d}^4q\,\frac{i}{q^2-m^2+i\epsilon}\Big(\delta^{(4)}(k_1+k_2-q)\,\delta^{(4)}(q-p_1-p_2) \\
&\qquad\qquad +\,\delta^{(4)}(k_1-p_1-q)\,\delta^{(4)}(q+k_2-p_2) \\
&\qquad\qquad +\,\delta^{(4)}(k_2-p_2-q)\,\delta^{(4)}(q+k_1-p_2) \\
&\qquad\qquad +\,\delta^{(4)}(k_1+k_2-q)\,\delta^{(4)}(q-p_1-p_2)\Big)\,.
\end{aligned}$$

One can see that, by integrating over q, all delta function pairs give $\delta^{(4)}(k_1+k_2-p_1-p_2)$, which is as expected from the conservation laws of energy and momentum. The evenness of the delta function, i.e. $\delta(x)=\delta(-x)$, results in the first and fourth, and second and third, deltas being equivalent. As a result, they combine to cancel the factor of 2! in front of the integral, hence

$$iT_{fi} = (-ig)^2\left(\frac{i}{(k_1+k_2)^2-m^2}+\frac{i}{(k_1-p_1)^2-m^2}\right)\underbrace{(2\pi)^4\,\delta^{(4)}(k_1+k_2-p_1-p_2)}_{\text{energy \& momentum conservation}}.$$

At this point, the **matrix element** of the process $\mathcal{M}_{fi}$ is defined as the non-conservation terms, i.e.

$$iT_{fi} = \mathcal{M}_{fi}\,(2\pi)^4\,\delta^{(4)}(k_1+k_2-p_1-p_2)\,,$$

where

$$\mathcal{M}_{fi} = (-ig)^2 \left(\frac{i}{(k_1 + k_2)^2 - m^2} + \frac{i}{(k_1 - p_1)^2 - m^2} \right).$$

3.4.4 Feynman Graphs and Rules

The above definition of the matrix element demonstrates two simultaneous processes for the scattering process $\chi^+(p_1)\chi^-(p_2) \to \chi^+(k_1)\chi^-(k_2)$, in that an intermediary real scalar field ϕ of mass m has two different momenta; $k_1 + k_2$ and $k_1 - p_1$. These processes are distinguished by labelling them according to whether the intermediary traverses space (s) or time (t), as shown in Figure 3.4.

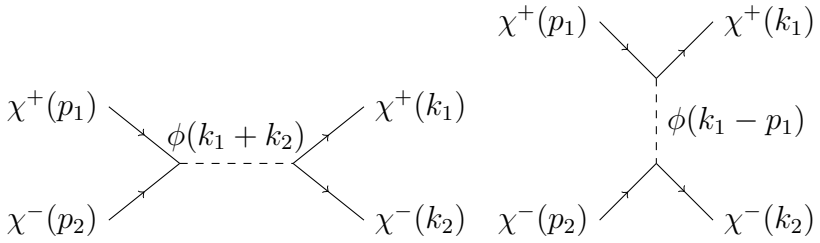


Figure 3.4: Left: Spatial (s) process.
Right: Temporal (t) process.

In scalar field theory, fields may self-interact, called a **quartic interaction**. This interaction is represented in the Lagrangian density by a potential term $-\frac{\lambda}{4!}\phi^4$. In the scattering process under discussion, this is then $-\frac{\lambda}{4!}\left(\chi^\dagger\chi\right)^2$. Including self-interactions, the Lagrangian density is hence

$$\mathcal{L} \mapsto \mathcal{L} + \left(-\frac{\lambda}{4!}\left(\chi^\dagger\chi\right)^2 \right).$$

The total Lagrangian density for scalar field theory is therefore

$$\mathcal{L} = \underbrace{\frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m_\phi^2\phi^2}_{\substack{\text{Klein-Gordon of real} \\ \text{scalar field } \phi}} + \underbrace{\frac{1}{2}\left(\partial_\mu\chi\right)\left(\partial^\mu\chi\right) - m_\chi^2\chi^\dagger\chi}_{\substack{\text{Klein-Gordon of complex} \\ \text{scalar fields } \chi^\pm}} - \underbrace{g\chi^\dagger\chi\phi}_{\text{interaction}} - \underbrace{\frac{\lambda}{4!}\left(\chi^\dagger\chi\right)^2}_{\substack{\text{self-} \\ \text{interaction}}}.$$

Each each element of a Feynman diagram has an intrinsic amplitude associated with it, hence one may then look at Feynman diagrams and calculate the transition amplitude of the process by inspection. The transition amplitudes for a number of elements is shown in Table 3.1.

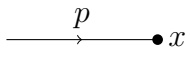
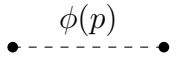
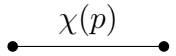
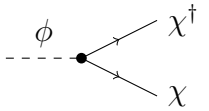
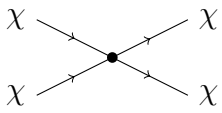
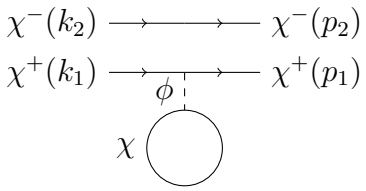
Element	**Diagram**	**Transition Amplitude,** iT_{fi}
External line		$e^{-ip\cdot x}$
ϕ-propagator		$\frac{i}{p^2-m_\phi^2+i\epsilon}$
χ-propagator		$\frac{i}{p^2-m_\chi^2+i\epsilon}$
ϕ-$\chi\chi$ vertex		$-ig$
$\chi\chi$-$\chi\chi$ vertex		$-i\lambda$
Loop		$\pm\int \frac{\mathrm{d}^4k}{(2\pi)^4}$,

Table 3.1: Transition amplitudes for elements of Feynman diagrams used in scalar field theory.

where the sign of the integral on loops depends on whether the loop is bosonic $(+)$ or fermionic $(-)$. The relation between the S-matrix element S_{fi} and $\mathcal{M}_{fi}$ for a scalar particle (spin-0 boson) is given by

$$S_{fi} = \mathbb{I} - T_{fi},$$

where T_{fi} in the momentum representation is proportional to $\mathcal{M}_{fi}$ by

$$T_{fi} \propto -i\,(2\pi)^4\,\delta^{(4)}\left(p_f - p_i\right)\mathcal{M}_{fi}.$$

To calculate the amplitude one therefore integrates over all momenta, which is aided by the inclusion of delta functions.

Steps to calculating amplitude of a given Feynman diagram

1. Write down $\delta^{(4)} = (2\pi)^4 \delta^{(4)}\left(p_f - p_i\right)$, such that it conserves momentum and energy at each internal line, then multiply together;
2. Write down one coupling constant for each vertex in the diagram;
3. Write down the propagator for each internal line;
4. Multiply results of steps 1-3 together;
5. Integrate over internal momenta;
6. For loops, add an additional integration factor of $\pm \int \frac{\mathrm{d}^4 k}{(2\pi)^4}$ for bosonic/fermionic loops.

3.5 Particle Decays and Cross Sections

3.5.1 Particle Decays

Consider the decay of particle $a\,(p)$ to n particles $b_n\,(k_n)$, as shown in Figure 3.5.

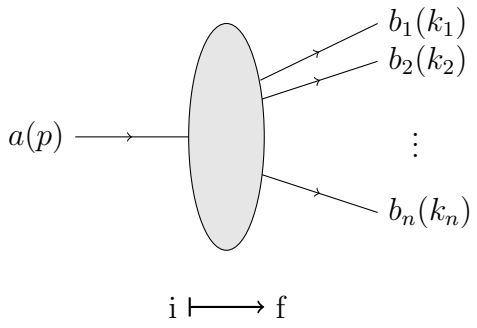


Figure 3.5: Decay of a single particle a to multiple particles b_n.

The decay rate of an unstable particle Γ is defined in its rest frame, i.e. where it is considered at rest. The decay rate of the above process is then given by

$$\mathrm{d}\Gamma_a = \frac{\mathrm{d}P_{a\to b_n}}{t} = \frac{\text{probability of decay}}{\text{unit time}}.$$

The probability of decay is the square of the transition amplitude, i.e.

$$\mathrm{d}P_{a\to b_n} = \left|T_{fi}\right|^2 = \Big((2\pi)^4\,\delta^{(4)}\left(k_1+k_2+\ldots+k_n-p\right)\Big)^2\left|\mathcal{M}_{fi}\right|^2,$$

and so the decay rate is given by

$$\mathrm{d}\Gamma_a = \left(\frac{1}{t}\right)\underbrace{\left(\frac{1}{2m_a}\right)}_{\substack{\text{normalisation}\\ \text{energy}}}\underbrace{\left(\frac{1}{V}\right)}_{\substack{\text{normalisation}\\ \text{volume}}}\underbrace{\Big((2\pi)^4\,\delta^{(4)}\left(k_1+k_2+\ldots+k_n-p\right)\Big)^2\left|\mathcal{M}_{fi}\right|^2}_{\mathrm{d}P_{a\to b_n}}$$

$$\times\ \underbrace{\frac{1}{s!}}_{\substack{\text{statistical}\\ \text{factor}}}\underbrace{\prod_{i=1}^{n}\frac{\mathrm{d}^3\mathbf{k}_i}{(2\pi)^3\,2E_{k_i}}}_{\substack{\text{phase space: sum}\\ \text{over all } |k_i\rangle \text{ states}}}.$$

The decay rate Γ_a has units of time^{-1} (s^{-1}) or energy (GeV). The reason for including the normalisation facots derives from an issue with having multiple delta functions:

$$\delta(x)\delta(x) = \delta(0)\delta(x) = \infty.$$

The phase space term also has a volume factor, but this may be removed by the phase space itself. Consider a particle in a box of length L. The solution to this is

$$\sin(k_n x) = \frac{e^{ik_n x} - e^{-ik_n x}}{2i},$$

where $k_n = \frac{2\pi n}{L}$, and hence $n = \frac{L}{2\pi}k$. Integrating over all harmonics then gives

$$\int \mathrm{d}^3 n = \frac{L^3}{(2\pi)^3}\underbrace{\int \mathrm{d}^3 k}_{=(2\pi)^3} = L^3 \equiv V.$$

This volume factor then cancels with that of the wavefunction.

Fermi's trick for renormalisation is to consider one of the delta functions as

an integral of an exponent over the four-volume Vt, i.e.

$$\left((2\pi)^4 \delta^{(4)}\left(k_1 + \ldots + k_n - p\right)\right)^2 = \int_{Vt} \underbrace{e^{i(k_1+\ldots k_n - p)\cdot x}}_{\substack{=1 \text{ when integral} \\ \text{is non-zero}}} (2\pi)^4 \delta^{(4)}\left(k_1 + \ldots + k_n - p\right)\ \mathrm{d}^4 x$$

$$= Vt \underbrace{(2\pi)^4 \delta^{(4)}\left(k_1 + \ldots + k_n - p\right)}_{\text{energy \& momentum conservation}}.$$

Using this trick for the squared delta functions in the decay rate equation;

$$\mathrm{d}\Gamma_a = \frac{1}{2m_a} (2\pi)^4 \delta^{(4)}\left(k_1 + \ldots + k_n - p\right) \left|\mathcal{M}_{fi}\right|^2 \frac{1}{s!} \prod_{i=1}^{n} \frac{\mathrm{d}^3\mathbf{k}_i}{(2\pi)^3\, 2E_{k_i}}.$$

Proof 3.8

The probability for the transition $|i\rangle \to |f\rangle$ in a time t is $P_{i\to f}(t)$, which is defined by

$$P_{i\to f}(t) = \left|\left\langle i \,\middle|\, U(t, t_0 \,\middle|\, f\right\rangle\right|^2.$$

Fermi's golden rule gives

$$\frac{\mathrm{d}P_{i\to f}}{\mathrm{d}t} = 2\pi \left|\langle n \,|\, \mathscr{H}_{\text{int}} \,|\, f\rangle\right|^2 \rho(E),$$

where $\rho(E)$ is the density of the final states. The motivation is to calculate the probability of a single particle state $|i\rangle$ becoming an n-particle final state $|f\rangle$, therefore denote $P_i = \sum_{j=1}^{n} P_j$. The probability of the state change is then

$$P_n = \frac{\left|\langle f \,|\, S \,|\, i\rangle\right|^2}{\langle i \,|\, i\rangle \langle f \,|\, f\rangle}$$

with $\langle i \,|\, i\rangle = (2\pi)^3\, 2E_{p_i}\delta^{(3)}(0)$ (from $\langle p \,|\, p\rangle = (2\pi)^3\, 2E_{p_i}\delta^{(3)}(\mathbf{p} - \mathbf{p})$). The delta function $\delta^{(3)}(0)$ may be replaced by the volume of the space V for correct normalisation, i.e.

$$\langle i \,|\, i\rangle = (2\pi)^3\, 2E_{p_i} V.$$

It is already known that

$$\begin{aligned}\langle f \,|\, S - \mathbb{I} \,|\, i\rangle &= i \left\langle f \,\middle|\, T_{fi} \,\middle|\, i \right\rangle \\ &= i \left(2\pi\right)^4 \delta^{(4)} \left(p_f - p_i\right) \left|\mathcal{M}_{fi}\right|^2 .\end{aligned}$$

By considering all final states, the probability of transition is

$$\begin{aligned}P_n &= \frac{1}{2mV} \prod_{j=1}^{n} \frac{1}{2E_{k_j} V} \left(\left(2\pi\right)^4 \delta^{(4)} \left(p_f - p_i\right) \right)^2 \left|\mathcal{M}_{fi}\right|^2 \\ &= \frac{1}{2m} \left(2\pi\right)^4 \delta^{(4)} \left(p_f - p_i\right) T \left|\mathcal{M}_{fi}\right|^2 \prod_{j=1}^{n} \frac{1}{2E_{k_j} V},\end{aligned}$$

where Fermi's trick has been used. The transition probability per unit time is then

$$\frac{P_n}{T} = \frac{1}{2m} \left(2\pi\right)^4 \delta^{(4)} \left(p_f - p_i\right) \left|\mathcal{M}_{fi}\right|^2 \prod_{j=1}^{n} \frac{1}{2E_{k_j} V}.$$

The decay rate is the probability per unit time of transition multiplied by the phase space volume, i.e. $\Gamma = V \frac{P_n}{T} \Rightarrow \mathrm{d}\Gamma_n = \frac{\mathrm{d}^3\mathbf{k}_i}{(2\pi)^3} \frac{P_n}{T}$, and so

$$\mathrm{d}\Gamma = \frac{1}{2m} \left(2\pi\right)^4 \delta^{(4)} \left(p_f - p_i\right) \left|\mathcal{M}_{fi}\right|^2 \frac{1}{s!} \prod_{j=1}^{n} \frac{\mathrm{d}^3\mathbf{k}_j}{\left(2\pi\right)^3 2E_{k_j}}.$$

3.5.2 Scattering Processes

Now consider a process in which two particles interacting, releasing n particles, as shown in Figure 3.6.

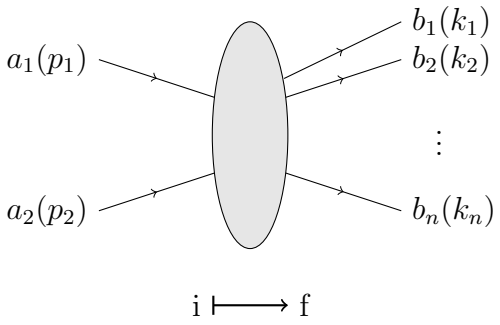


Figure 3.6: Two-particle interaction, producing n particles.

3.5.3 Cross Sections

Definition 3.1: Cross Section

The **cross section**, denoted σ, is defined as the ratio of the transition rate $\frac{P_n}{t}$ to the **flux** of incoming particles F, i.e.

$$\sigma \coloneqq \frac{\left(\frac{P_n}{t}\right)}{F}.$$

Its units are hence length^{-2} or energy^2, with conventional units for particle interactions being *barns*, with $1\,\mathrm{b} = 10^{-24}\,\mathrm{cm}^{-2}$.

Definition 3.2: Flux

The flux F is the rate at which particles pass through a given area per unit time, i.e.

$$F = \frac{1 \text{ particle}}{\Delta y\ \Delta x\ \Delta t} = \frac{\left(\frac{\Delta x}{\Delta t}\right)}{\Delta x\ \Delta y\ \Delta z} = \frac{\Delta v}{V} = \frac{|\mathbf{v}_1 - \mathbf{v}_2|}{V},$$

where $|\mathbf{v}_1 - \mathbf{v}_2|$ is the relative velocity of incoming particles and V is the volume through which the particles pass.

The rate of decay for the two-particle interaction in Figure 3.6 is given by

$$\mathrm{d}\Gamma = \frac{\mathrm{d}P_n}{t} = \frac{1}{2E_{p_1}}\frac{1}{2E_{p_2}}\left(\frac{1}{V}\right)^2 \frac{1}{t}\left((2\pi)^4\,\delta^{(4)}\left(k_1 + \ldots + k_n - p_1 - p_2\right)\right)^2 \times \left|\mathcal{M}_{fi}\right|^2 \prod_{i=1}^{n} \frac{\mathrm{d}^3\mathbf{k}_i}{(2\pi)^3\,2E_{k_i}}.$$

The cross section for the interaction is therefore

$$\mathrm{d}\sigma = \frac{\mathrm{d}\Gamma}{F} = \frac{1}{|\mathbf{v}_1 - \mathbf{v}_2|}\frac{1}{2E_{p_1}}\frac{1}{2E_{p_2}}\left|\mathcal{M}_{fi}\right|^2 (2\pi)^4\,\delta^{(4)}\left(p_f - p_i\right)\prod_{i=1}^{n}\frac{\mathrm{d}^3\mathbf{k}_i}{(2\pi)^3\,2E_{k_i}}.$$

However, the true cross section σ_{true} must remove double-counting, which introduces a factor of $\frac{1}{n!} = S$, which is the statistical factor that was introduced earlier, and so

$$\mathrm{d}\sigma_{\text{true}} = S\ \mathrm{d}\sigma.$$

The number of scattering events N is given by $N = \sigma_{\text{true}}L$, where L is the luminosity of the collision.

3.6 Kinematics

In a centre-of-mass (CoM) system, a decaying particle a with four-momentum p may be considered at rest. As such, its four-momentum is

$$p^\mu = (m_a, \mathbf{0})\,,$$

i.e. $p^2 = m_a^2$. Consider a particle a with momentum p decaying into two particles b_1 and b_2 with respective momenta k_1 and k_2, as shown in Figure 3.7.

The four-momenta of the daughter particles k_1 and k_2 are denoted

$$k_1 = (E_1, \mathbf{k}_1)$$
$$k_2 = (E_2, \mathbf{k}_2)\,,$$

such that $k_1^2 = E_1^2 - \mathbf{k}_1^2 = m_1^2$ and $k_2^2 = E_2^2 - \mathbf{k}_2^2 = m_2^2$. By conservation of momentum and energy

$$p^\mu \overset{!}{=} k_1^\mu + k_2^\mu,$$

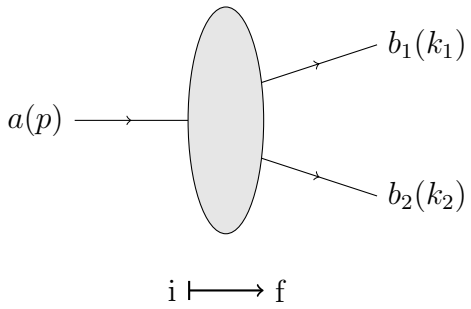


Figure 3.7: Decay of a single particle a to two particles b_1 and b_2.

which in the CoM frame then gives the zeroth components as $2m_a E_i = 2p\cdot k_i$, and hence

$$E_i = \frac{1}{2m_a} 2p \cdot k_i.$$

From expanding $(p - k_i)^2$, one gets the right-hand side of the above as

$$2p \cdot k_i = p^2 + k_i^2 - (p - k_i)^2 ,$$

where one denotes $(p - k_i)^2$ as k_j^2, the squared four-momentum of the other daughter particle. The cross term is then $2p\cdot k_i = p^2 + k_i^2 - k_j^2 = m_a^2 + k_i^2 - k_j^2$. The energy of each daughter particle is therefore given by

$$E_i = \frac{1}{2m_a}\left(m_a^2 + m_i^2 - m_j^2\right).$$

One can check conservation of energy:

$$\begin{aligned} E_1 + E_2 &= \frac{1}{2m_a}\left(m_a^2 + m_1^2 - m_2^2\right) + \frac{1}{2m_a}\left(m_a^2 - m_1^2 + m_2^2\right) \\ &= \frac{1}{2m_a}\left(2m_a^2\right) \\ &= m_a = E_a. \end{aligned}$$

As momentum must also be conserved, $\mathbf{p} = \mathbf{k}_1 + \mathbf{k}_2 = 0$ and hence $\mathbf{k}_1 = -\mathbf{k}_2$.

The Källén function $\lambda(x, y, z)$ is useful for formulating individual particle momenta in three-body decays. The function is defined by

$$\begin{aligned} \lambda\left(x^2, y^2, z^2\right) &\coloneqq (x + y + z)(x - y + z)(x + y - z)(x - y - z) \\ &= \left(x^2 - y^2 - z^2\right)^2 - 4y^2z^2 \end{aligned}$$

or

$$\begin{aligned}\lambda(x,y,z) &\coloneqq (x-y-z)^2 - 4yz \\ &= \left(x^2-y^2-z^2\right) - 2xy - 2xz - 2yz.\end{aligned}$$

This function is invariant under change of variable order, i.e. $\lambda(z,x,y) = \lambda(x,y,z)$ and as such this function is useful in defining the four-momenta in three-body decays. For example, reconsider the earlier three-body decay

$$\begin{aligned}|k_1| = \sqrt{k_1^2} &= \sqrt{E_1^2 - m_1^2} \\ &= \sqrt{\left(\frac{1}{2m_a}\left(m_a^2 + m_1^2 - m_2^2\right)\right)^2 - m_1^2} \\ &= \sqrt{\frac{1}{4m_a^2}\left(m_a^4 + m_1^4 + 2m_a^2m_1^2 - 2m_am_2^2 - 2m_1^2m_2^2\right) - \frac{4m_a^2m_1^2}{4m_a^2}} \\ &= \frac{1}{2m_a}\sqrt{m_a^4 + m_1^4 + m_2^4 - 2m_a^2m_1^2 - 2m_am_2^2 - 2m_1^2m_2^2} \\ &= \frac{1}{2m_a}\lambda^{\frac{1}{2}}\left(m_a^2, m_1^2, m_2^2\right).\end{aligned}$$

By similar methodology, one can show that $|k_1| = |k_2|$.

3.6.1 Decay Width for $a \mapsto b_1 b_2$

The decay rate for a decay $a \to b_1 b_2$, e.g. $\phi \to \chi^+\chi^-$, has been derived as

$$\Gamma_a = \frac{1}{2m_a}\int_{\mathbf{k}_1}\int_{\mathbf{k}_2}\left|\mathcal{M}_{fi}\right|^2(2\pi)^4\,\delta^{(4)}\left(k_1+k_2-p\right).$$

As the matrix element is a constant of the interaction, it may be brought in front of the integrals, while the factors of 2π from the integrals cancel with those inside the integrand. As a result of this and the evenness of the delta function, the decay rate is then

$$\Gamma_a = \frac{1}{2m_a}\frac{\left|\mathcal{M}_{fi}\right|^2}{(2\pi)^2}\int\frac{\mathrm{d}^3\mathbf{k}_1}{2E_1}\int\frac{\mathrm{d}^3\mathbf{k}_2}{2E_2}\delta^{(4)}\left(p-k_1-k_2\right).$$

The three-integrals can be converted into four-integrals with a delta function, i.e.

$$\int \frac{\mathrm{d}^3\mathbf{k}_i}{2E_i} = \int \mathrm{d}^4k_i \; \delta\left(k_i^2 - m_i^2\right).$$

Proof 3.9: Three-integrals to four-integrals

Consider an integral over a delta function $\int \delta(f(x)) \; \mathrm{d}x$ and let $y = f(x)$. As such, $\mathrm{d}y = \frac{\mathrm{d}f(x)}{\mathrm{d}x} \; \mathrm{d}x = f'(x) \; \mathrm{d}x$, hence $\mathrm{d}x = \frac{1}{\left|f'(x)\right|} \; \mathrm{d}y$, and so

$$\int \delta(f(x)) \; \mathrm{d}x = \int \frac{1}{\left|f'(x)\right|} \delta(y) \; \mathrm{d}y.$$

Now consider the integral $\int \delta(k_1^2 - m_1^2) \; \mathrm{d}E_1 = \int \delta(E_1^2 - \mathbf{k}_1^2 - m_1^2) \; \mathrm{d}E_1$ and let $y = E_1^2 - \mathbf{k}_1^2 - m_1^2$, then $\mathrm{d}y = 2E_1 \; \mathrm{d}E_1 \Rightarrow \mathrm{d}E_1 = \frac{1}{|2E_1|} \; \mathrm{d}y$. The integral is then

$$\int \delta(k_1^2 - m_1^2) \; \mathrm{d}E_1 = \int \frac{1}{|2E_1|} \delta(y) \; \mathrm{d}y = \frac{1}{|2E_1|},$$

and hence

$$\int \frac{\mathrm{d}^3\mathbf{k}_1}{2E_1} = \int \mathrm{d}^3\mathbf{k}_1 \int \delta(k_1^2 - m_1^2) \; \mathrm{d}E_1 = \int \mathrm{d}^4k_1 \; \delta(k_1^2 - m_1^2).$$

The decay rate is therefore

$$\Gamma_a = \frac{1}{2m_a} \frac{\left|\mathcal{M}_{fi}\right|^2}{(2\pi)^2} \int \mathrm{d}^4k_1 \int \mathrm{d}^4k_2 \; \delta\left(k_1^2 - m_1^2\right) \; \delta\left(k_2^2 - m_2^2\right) \; \delta^{(4)}\left(p - k_1 - k_2\right).$$

Integrating out k_2 then gives

$$\begin{aligned}\Gamma_a &= \frac{1}{2m_a} \frac{\left|\mathcal{M}_{fi}\right|^2}{(2\pi)^2} \int \mathrm{d}^4k_1 \; \delta\left(k_1^2 - m_1^2\right) \; \delta^{(4)}\left(\left(p - k_1\right)^2 - m_2^2\right) \\ &= \frac{1}{2m_a} \frac{\left|\mathcal{M}_{fi}\right|^2}{(2\pi)^2} \int \frac{\mathrm{d}^3\mathbf{k}_1}{2E_1} \; \delta\left(p^2 - 2p \cdot k_1 + m_1^2 - m_2^2\right).\end{aligned}$$

In the CoM frame $p = (m_a, \mathbf{0})$, and so $p \cdot k_1 = m_a E_1$. The above delta function is hence $\delta\left(p^2 - 2p \cdot k_1 + m_1^2 - m_2^2\right) = \delta\left(m_a^2 - 2m_a E_1 + m_1^2 - m_2^2\right)$.

From the identity $\delta(ax) = \frac{1}{|a|}\delta(x)$, then

$$\delta\left(p^2 - 2p\cdot k_1 + m_1^2 - m_2^2\right) = \frac{1}{2m_a}\delta\left(E_1 - \frac{m_a^2 + m_1^2 - m_2^2}{2m_a}\right).$$

The decay rate is then

$$\Gamma_a = \frac{1}{2m_a}\frac{\left|\mathcal{M}_{fi}\right|^2}{(2\pi)^2}\int \frac{\mathrm{d}^3\mathbf{k}_1}{2E_1}\,\frac{1}{2m_a}\delta\left(E_1 - \frac{m_a^2 + m_1^2 - m_2^2}{2m_a}\right),$$

where $\frac{m_a^2+m_1^2-m_2^2}{2m_a}$ is the energy in the centre-of-mass frame, often denoted E_1^*. In spherical polar coordinates $\mathrm{d}^3\mathbf{k}_1 = |k_1|^2\ \mathrm{d}k_1\ \mathrm{d}\Omega = 4\pi\,|k_1|^2\ \mathrm{d}k_1$. Furthermore, $E_1^2 = \mathbf{k}_1 - m_1^2 \Rightarrow 2E_1\ \mathrm{d}E_1 = 2k_1\ \mathrm{d}k_1$, and so

$$\begin{aligned}\mathrm{d}^3\mathbf{k}_1 &= 4\pi\,|k_1|\,E_1\ \mathrm{d}E_1\\ \Rightarrow \int\frac{\mathrm{d}^3\mathbf{k}_1}{2E_1} &= \int\frac{4\pi\,|k_1|\ \mathrm{d}E_1}{2}.\end{aligned}$$

The decay rate is finally given by

$$\begin{aligned}\Gamma_a &= \frac{1}{2m_a}\frac{\left|\mathcal{M}_{fi}\right|^2}{(2\pi)^2}\int\frac{4\pi\,|k_1|\ \mathrm{d}E_1}{2}\,\frac{1}{2m_a}\delta\left(E_1 - \frac{m_a^2 + m_1^2 - m_2^2}{2m_a}\right)\\ &= \frac{1}{8m_a^2}\frac{1}{\pi}\left|\mathcal{M}_{fi}\right|^2\int \underbrace{|k_1|}_{=\frac{1}{2m_a}\lambda^{\frac{1}{2}}}\underbrace{\delta\left(E_1 - \frac{m_a^2 + m_1^2 - m_2^2}{2m_a}\right)}_{\text{used for Källén in } k_1}\mathrm{d}E_1\\ &= \frac{\left|\mathcal{M}_{fi}\right|^2}{16\pi m_a^3}\lambda^{\frac{1}{2}}\left(m_a^2, m_1^2, m_2^2\right).\end{aligned}$$

3.6.2 Kinematics for $2 \mapsto 2$ scattering

Consider the $2 \mapsto 2$ scattering process shown in Figure 3.8.

By conservation laws:

$$\begin{aligned}p_1 + p_2 &= k_1 + k_2\\ p_1^2 = m_{a_1}^2 \qquad & p_2^2 = m_{a_2}^2\\ k_1^2 = m_{b_1}^2 \qquad & k_2^2 = m_{b_2}^2.\end{aligned}$$

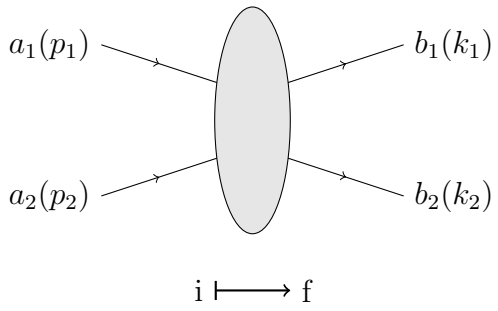


Figure 3.8: Scattering process of two complex scalar fields.

One can now define the Lorentz-invariant Mandelstam variables s, t and u

$$s \coloneqq \left(\sum_n p_{i_n}\right)^2$$

which for

$$\begin{aligned} s &= (p_1 + p_2)^2 = (k_1 + k_2)^2 = m_{a_1}^2 + 2E_{a_1}E_{a_2} - 2\mathbf{p}_1 \cdot \mathbf{p}_2 + m_{a_2}^2 \\ t &= (p_1 - k_1)^2 = (p_2 - k_2)^2 = m_{a_1}^2 - 2E_{a_1}E_{b_1} - 2\mathbf{p}_1 \cdot \mathbf{k}_1 + m_{b_1}^2 \\ u &= (p_1 - k_2)^2 = (p_2 - k_1)^2 = m_{a_1}^2 + 2E_{a_1}E_{b_2} - 2\mathbf{p}_1 \cdot \mathbf{k}_2 + m_{a_2}^2, \end{aligned}$$

which then satisfy

$$s + t + u = m_{a_1}^2 + m_{a_2}^2 + m_{b_1}^2 + m_{b_2}^2.$$

Previously derived equations may now be written in terms of these new variables:

$$|p_1| = |p_2| = \frac{\lambda^{\frac{1}{2}}\left(s, m_{a_1}^2, m_{a_2}^2\right)}{2\sqrt{s}}$$

$$|k_1| = |k_2| = \frac{\lambda^{\frac{1}{2}}\left(s, m_{b_1}^2, m_{b_2}^2\right)}{2\sqrt{s}}$$

$$E_{a_1}^2 = \frac{s + m_{a_1}^2 - m_{a_2}^2}{2\sqrt{s}} \qquad E_{a_2}^2 = \frac{s - m_{a_1}^2 + m_{a_2}^2}{2\sqrt{s}}$$

$$E_{b_1}^2 = \frac{s + m_{b_1}^2 - m_{b_2}^2}{2\sqrt{s}} \qquad E_{b_2}^2 = \frac{s - m_{b_1}^2 + m_{b_2}^2}{2\sqrt{s}}.$$

The energies are analogous to the $1 \to 2$ kinematics, with $s = m_a^2$, i.e. $\sqrt{s} = E$. The is akin to treating the incoming ensemble as a single particle.

3.6.3 Differential Cross Section

Now consider the cross section of the interaction within $2 \mapsto 2$ kinematics.

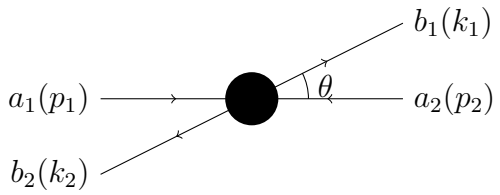


Figure 3.9: $2 \mapsto 2$ interaction.

The differential cross section quantifies the intrinsic rate of an event occurring at a given angle and is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1} = \frac{1}{64\pi^2 s}\frac{|k_1^*|}{|p_1^*|}\left|\mathcal{M}_{fi}\right|^2,$$

where $\mathrm{d}\Omega_1 = \sin(\theta_1)\ \mathrm{d}\theta_1\ \mathrm{d}\phi_1$. From this, one can also derive the rate at which collisions occur as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\left|\mathcal{M}_{fi}\right|^2}{16\pi\lambda\left(s, m_{a_1}^2, m_{a_2}^2\right)},$$

where $t = \frac{|p_1||k_1|}{\pi}\ \mathrm{d}\Omega$.

Proof 3.10: Cross section differentials

For ease of notation, refer to Figure 3.10 instead of Figure 3.9.

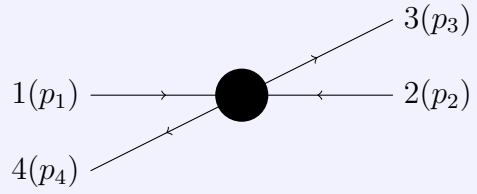


Figure 3.10: Simplified $2 \mapsto 2$ interaction.

The relativistic form of Fermi's golden rule is given by

$$2E_1 \cdot 2E_2 v_{\text{rel}}\, \mathrm{d}\sigma = (2\pi)^4\, \delta^{(4)}\left(p_f - p_i\right) \left|\mathcal{M}_{fi}\right|^2 \times \frac{\mathrm{d}^4 p_3}{(2\pi)^4} 2\pi\delta\left(p_3^2 - m_3^2\right) \cdot \frac{\mathrm{d}^4 p_3}{(2\pi)^4} 2\pi\delta\left(p_4^2 - m_4^2\right).$$

It has previously been shown that

$$\int \frac{\mathrm{d}^3 p_3}{(2\pi)^3\, 2E_3} = \frac{1}{(2\pi)^3} \int \mathrm{d}^4 p_3\; \delta\left(p_3^2 - m_3^2\right),$$

which then gives the Fermi golden rule as

$$\begin{aligned} 2E_1 \cdot 2E_2 v_{\text{rel}}\, \mathrm{d}\sigma &= (2\pi)^4\, \delta^{(4)}\left(p_f - p_i\right) \left|\mathcal{M}_{fi}\right|^2 \frac{\mathrm{d}^3 p_3}{(2\pi)^3\, 2E_3} \frac{\mathrm{d}^3 p_4}{(2\pi)^3\, 2E_4} \\ &= 2\pi\delta^{(4)}\left(p_f - p_i\right) \left|\mathcal{M}_{fi}\right|^2 \frac{\mathrm{d}^3 p_3}{(2\pi)^3\, 2E_3} \frac{\mathrm{d}^3 p_4}{2E_4}. \end{aligned}$$

Integrating out $\mathrm{d}^3 p_4$ using the delta function leaves only energy terms:

$$2E_1 \cdot 2E_2 v_{\text{rel}}\, \mathrm{d}\sigma = 2\pi\delta\left(E_f - E_i\right) \left|\mathcal{M}_{fi}\right|^2 \frac{\mathrm{d}^3 p_3}{(2\pi)^3\, 2E_3} \frac{1}{2E_4}.$$

Changing to spherical polar coordinates alters $\mathrm{d}^3 p_3$ as $\mathrm{d}^3 p_3 = \left|p_3\right|^2\, \mathrm{d}p_3\, \mathrm{d}\Omega_1$. The remaining delta function can be integrated out by using the following relation:

$$\begin{aligned} \int \delta(f(x))\, \mathrm{d}x &= \int \delta(y) \frac{\mathrm{d}y}{|f'|} = \frac{1}{|f'|} \\ \Rightarrow \int \delta(E_f - E_i)\, \mathrm{d}p_3 &= \int \delta(p_f^0 - p_i^0)\, \mathrm{d}E \\ &= \left(\frac{1}{|p_3'|}\right)^{-1} \\ &= \left(\frac{\mathrm{d}\left(E_f - E_i\right)}{\mathrm{d}|\mathbf{p}_3|}\right)^{-1}. \end{aligned}$$

The golden rule is now

$$2E_1 \cdot 2E_2 v_{\text{rel}} \, \mathrm{d}\sigma = 2\pi \left(\frac{\mathrm{d}\left(E_f - E_i\right)}{\mathrm{d}|\mathbf{p}_3|} \right)^{-1} \left|\mathcal{M}_{fi}\right|^2 \frac{\mathrm{d}^3 p_3}{(2\pi)^3 \, 2E_3} \frac{1}{2E_4} \, \mathrm{d}\Omega_1.$$

In the CoM frame, $\mathbf{p}_1 = -\mathbf{p}_2 \Rightarrow |\mathbf{p}_1| + |\mathbf{p}_2|$ and $\mathbf{p}_3 = -\mathbf{p}_4 \Rightarrow |\mathbf{p}_3| + |\mathbf{p}_4|$. The difference in energy is then

$$\begin{aligned} E_f - E_i &= \left(\sqrt{\mathbf{p}_3^2 + m_3^2} + \sqrt{\mathbf{p}_4^2 + m_4^2} \right) - E_i \\ &= \left(\sqrt{\mathbf{p}_3^2 + m_3^2} + \sqrt{\mathbf{p}_3^2 + m_4^2} \right) - E_i. \end{aligned}$$

As such, the differential of energy with respect to momentum is

$$\begin{aligned} \frac{\mathrm{d}\left(E_f - E_i\right)}{\mathrm{d}|\mathbf{p}_3|} &= \frac{|\mathbf{p}_3|}{\sqrt{\mathbf{p}_3^2 + m_3^2}} + \frac{|\mathbf{p}_3|}{\sqrt{\mathbf{p}_3^2 + m_4^2}} \\ &= \frac{|\mathbf{p}_3|}{E_3} + \frac{|\mathbf{p}_3|}{E_4} \\ &= \frac{|\mathbf{p}_3| \left(E_3 + E_4\right)}{E_3 E_4} \\ &= \frac{|\mathbf{p}_3| \sqrt{s}}{E_3 E_4}, \end{aligned}$$

where $\sqrt{s} = E_1 + E_2 = E_3 + E_4$. The relative velocity v_{rel} of particles 1 and 2 is the difference in momenta

$$v_{\text{rel}} = |v_1 - v_2| = |v_1| + |v_2| = \frac{|\mathbf{p}_1|}{E_1} + \frac{|\mathbf{p}_2|}{E_2} = \frac{|\mathbf{p}_1| \left(E_1 + E_2\right)}{E_1 E_2} = \frac{|\mathbf{p}_1| \sqrt{s}}{E_1 E_2}.$$

Considering the sides of Fermi's golden rule separately;

$$2E_1 2E_2 v_{\text{rel}} \, \mathrm{d}\sigma = 2E_1 2E_2 \frac{|\mathbf{p}_1| \sqrt{s}}{E_1 E_2} \, \mathrm{d}\sigma = 4 \, |\mathbf{p}_1| \sqrt{s} \, \mathrm{d}\sigma$$

$$2\pi \left(\frac{\mathrm{d}\left(E_f - E_i\right)}{\mathrm{d}|\mathbf{p}_3|} \right)^{-1} \left|\mathcal{M}_{fi}\right|^2 \frac{\mathrm{d}^3 p_3}{(2\pi)^3 2E_3} \frac{1}{2E_4}\, \mathrm{d}\Omega_1$$

$$= 2\pi \frac{E_3 E_4}{|\mathbf{p}_3|\sqrt{s}} \left|\mathcal{M}_{fi}\right|^2 \frac{\mathrm{d}^3 p_3}{(2\pi)^3 2E_3 2E_4}\, \mathrm{d}\Omega_1$$

$$= \frac{|\mathbf{p}_3|}{4(2\pi)^2 \sqrt{s}} \left|\mathcal{M}_{fi}\right|^2\, \mathrm{d}\Omega_1,$$

and hence

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1} = \frac{\left|\mathcal{M}_{fi}\right|^2}{64\pi^2 s} \frac{|\mathbf{p}_3|}{|\mathbf{p}_1|}.$$

Using this result with the relations $|\mathbf{p}_3| = |\mathbf{k}_3|$ and $\mathrm{d}t = \frac{|\mathbf{p}_1||\mathbf{k}_1|}{\pi}\, \mathrm{d}\Omega_1$:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1} \frac{\pi}{|\mathbf{p}_1|\,|\mathbf{k}_1|}$$
$$= \frac{1}{64\pi s} \frac{1}{p_1^2} \left|\mathcal{M}_{fi}\right|^2.$$

It is already known that $|\mathbf{k}_1| = \frac{1}{2m_a}\lambda^{\frac{1}{2}}\left(m_a^2, m_1^2, m_2^2\right)$, and so from $|\mathbf{p}_1| = |\mathbf{k}_1|$ one gets $p_1^2 = \frac{1}{4m_a^2}\lambda\left(m_a^2, m_1^2, m_2^2\right) = \frac{1}{4s}\lambda\left(s, m_1^2, m_2^2\right)$, thus

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi} \frac{1}{\lambda\left(s, m_1^2, m_2^2\right)} \left|\mathcal{M}_{fi}\right|^2.$$

3.7 Unitarity of the S-Matrix Operator, $\hat{S}$

By construction, the S-matrix is unitary, i.e. $S^\dagger S = S S^\dagger = \mathbb{I}$, where

$$\mathbb{I} = |0\rangle\langle 0| + \int_{\mathbf{k}_1} |k_1\rangle\langle k_1| + \int_{\mathbf{k}_1}\int_{\mathbf{k}_2} |k_1, k_2\rangle\langle k_1, k_2| + \ldots$$

From the earlier relation to the transition amplitude $S = \mathbb{I} + iT$;

$$\begin{aligned} S^\dagger S &= (\mathbb{I} + iT)^\dagger (\mathbb{I} + iT) \\ &= \mathbb{I} + i\left(T - T^\dagger\right) + T^\dagger T \\ &\overset{!}{=} \mathbb{I}. \end{aligned}$$

As a result, $\frac{1}{2i}\left(T - T^\dagger\right) = \frac{1}{2}T^\dagger T$. The relation between different initial states is given as

$$\begin{aligned} \left\langle i' \,\middle|\, i \right\rangle = \left\langle i' \,\middle|\, S^\dagger S \,\middle|\, i \right\rangle &= \sum_f \left\langle i' \,\middle|\, S \,\middle|\, f \right\rangle^\dagger \left\langle f \,\middle|\, S \,\middle|\, i \right\rangle \\ &= \sum_f \left\langle f \,\middle|\, S \,\middle|\, i' \right\rangle^* \left\langle f \,\middle|\, S \,\middle|\, i \right\rangle, \end{aligned}$$

and hence the transmissions are

$$\frac{1}{2i}\left(\left\langle i' \,\middle|\, T \,\middle|\, i \right\rangle - \left\langle i \,\middle|\, T \,\middle|\, i' \right\rangle^*\right) = \frac{1}{2}\sum_f \left\langle f \,\middle|\, T \,\middle|\, i' \right\rangle^* \left\langle f \,\middle|\, T \,\middle|\, i \right\rangle.$$

As the transition amplitude elements T_{fi} are defined by

$$T_{fi} \coloneqq \langle f \,|\, T \,|\, i \rangle \equiv (2\pi)^4 \, \delta^{(4)}\left(p_i - k_f\right) \mathcal{M}_{fi},$$

then for $i = i'$

$$\begin{aligned} \frac{1}{2i}\left(\langle i \,|\, T \,|\, i \rangle - \langle i \,|\, T \,|\, i \rangle^*\right) &= \frac{1}{2}\sum_f \langle f \,|\, T \,|\, i \rangle^* \langle f \,|\, T \,|\, i \rangle \\ \frac{1}{2i}\left(\mathcal{M}_{ii} - \mathcal{M}_{ii}^*\right) &= \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}_f}{(2\pi)^3 \, 2E_f} (2\pi)^4 \, \delta^{(4)}\left(k_i - p_f\right) \left|\mathcal{M}_{fi}\right|^2. \end{aligned}$$

For any complex number $z = x + iy$, the difference between itself and its complex conjugate is then

$$z - z^* = \big(\mathrm{Re}(z) + \mathrm{Im}(z)\big) - \big(\mathrm{Re}(z) - \mathrm{Im}(z)\big) = 2\,\mathrm{Im}(z),$$

and hence the **forward scattering** $\mathrm{Im}\left(\mathcal{M}_{ii}\right)$ is given by

$$\begin{aligned} \mathrm{Im}\left(\mathcal{M}_{ii}\right) &= \frac{i}{2} \underbrace{\int \frac{\mathrm{d}^3\mathbf{p}_f}{(2\pi)^3 \, 2E_f} (2\pi)^4 \, \delta^{(4)}\left(k_i - p_f\right) \left|\mathcal{M}_{fi}\right|^2}_{= 2\lambda^{\frac{1}{2}}\left(s, m_{a_1}^2, m_{a_2}^2\right)\sigma(a_1 a_2 \mapsto f)} \\ &= i\lambda^{\frac{1}{2}}\left(s, m_{a_1}^2, m_{a_2}^2\right) \sigma\left(a_1 a_2 \mapsto f\right). \end{aligned}$$

3.8 The Optical Theorem

The optical theorem is a general law of wave scattering theory, which relates the forward scattering amplitude to the total cross section of the scatterer:

$$\sigma_{\text{tot}} = \frac{4\pi}{k} \operatorname{Im}\left(f\left(0\right)\right),$$

where $f\left(0\right)$ is the scattering amplitude at an angle of zero. For $i = i'$, where $|i\rangle = \left|a_1(p_1), a_2(p_2)\right\rangle$;

$$\operatorname{Im}\left(\mathcal{M}_{ii}\right) = \lambda^{\frac{1}{2}}\left(s, m_{a_1}^2, m_{a_2}^2\right) \sigma\left(a_1 a_2 \mapsto f\right),$$

where f indicates all possible final states that come from a scattering of particles a_1 and a_2.

Chapter 4

Quantum Electrodynamics

4.1 Spinors

Definition 4.1: Spinors

Mathematically speaking, **spinors** are elements of a complex vector space that can be associated with Euclidean space. This is akin to an stating that spinors are physical objects with rotation.

4.1.1 Weyl Spinors

Weyl spinors, denoted $\xi_\alpha(x)$ and $\overline{\eta}^{\dot{\alpha}}(x)$, where the overbar and over-dot represent complex conjugation, and $\alpha, \dot{\alpha} = 1, 2$ are two-dimensional complex vectors whose components are Grassmann numbers, i.e. $\xi_\alpha, \overline{\eta}^{\dot{\alpha}} \in \mathbb{G}$. As a result, these spinors are anti-commutative, i.e. $xy = -yx$, and hence

$$\begin{aligned}
\xi_i \cdot \xi_j &= -\xi_j \cdot \xi_i \\
\overline{\eta}^{\dot{\alpha}} \cdot \overline{\eta}^{\dot{\beta}} &= -\overline{\eta}^{\dot{\beta}} \cdot \overline{\eta}^{\dot{\alpha}} \\
\overline{\eta}^{\dot{\alpha}} \cdot \xi_j &= -\xi_j \cdot \overline{\eta}^{\dot{\alpha}} \\
\xi_i \cdot \xi_i &= -\xi_i \cdot \xi_i \Rightarrow \overset{!}{=} 0.
\end{aligned}$$

Under a Lorentz transformation ($x^\mu \mapsto x'^\mu = \Lambda^\mu_\nu x^\nu; \Lambda \in \mathrm{SO}(1,3)$), the spinors are transformed as

$$\begin{aligned}
\xi_\alpha(x) &\mapsto \xi'_\alpha(x') = M^\beta_\alpha \xi_\beta(x) \\
\xi^\alpha(x) &\mapsto \xi'^\alpha(x') = \left(M^{-1}\right)^\alpha_\beta \xi^\beta(x) \\
\overline{\eta}_{\dot{\alpha}}(x) &\mapsto \overline{\eta}'_{\dot{\alpha}}(x') = \left(M^\dagger\right)^{\dot{\beta}}_{\dot{\alpha}} \overline{\eta}_{\dot{\beta}}(x) \\
\overline{\eta}^{\dot{\alpha}}(x) &\mapsto \overline{\eta}'^{\dot{\alpha}}(x') = \left(\left(M^\dagger\right)^{-1}\right)^{\dot{\alpha}}_{\dot{\beta}} \overline{\eta}^{\dot{\beta}}(x),
\end{aligned}$$

thus there are four independent vectors spaces involved. M is a member of the 2D complex special linear group, i.e. $M \in \mathrm{SL}(2,\mathbb{C})$, and is the comlexification of SU: $e^{i\theta^i \cdot \frac{\sigma_i}{2}}$, i.e.

$$M = e^{i\left(\theta^i + i\phi^i\right)\cdot\frac{\sigma_i}{2}}.$$

This has determinant $\det(M) = 1$ and there exists two independent Lorentz transformations:

$$\begin{aligned}
M &\to \left(\frac{1}{2}, 0\right); \\
M^\dagger &\to \left(0, \frac{1}{2}\right),
\end{aligned}$$

which relate to left- and right-hand spinor spaces, respectively, i.e. $\xi_\alpha \in \left(\frac{1}{2}, 0\right) =$ left-handed; $\overline{\eta}_{\dot{\alpha}} \in \left(0, \frac{1}{2}\right) =$ right-handed.

Example 4.1: SO(2)

An example of SO(2) is a planar rotation, as shown in Figure 4.1.

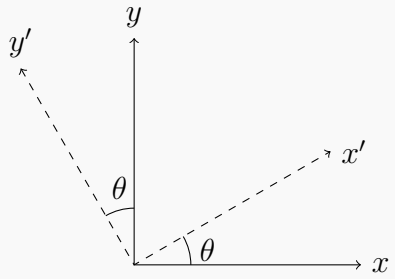


Figure 4.1

The transformation from the initial plane to the rotated plane is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The diagonalised form of this is then

$$\begin{pmatrix} \frac{x'+iy'}{\sqrt{2}} \\ \frac{x'-iy'}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} \frac{x+iy}{\sqrt{2}} \\ \frac{x-iy}{\sqrt{2}} \end{pmatrix}.$$

By defining a complex number $z = x + iy$, and hence $z' = x' + iy'$, the diagonalised equation becomes

$$\begin{pmatrix} z' \\ \overline{z}' \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} z \\ \overline{z} \end{pmatrix}.$$

As a result, the 2D special orthogonal group is $\mathrm{SO}(2) \approx U(1) \oplus \overline{U}(1)$, and so

$$\begin{aligned} z &\mapsto re^{i\theta}; \\ \overline{z} &\mapsto re^{-i\theta}. \end{aligned}$$

Some duality relations between Weyl spinors are:

$$\begin{aligned} (\xi^\alpha)^\dagger &= \overline{\xi}^{\dot{\alpha}}; \qquad (\eta^\alpha)^\dagger = \overline{\eta}^{\dot{\alpha}} \\ (\xi_\alpha)^\dagger &= \overline{\xi}_{\dot{\alpha}}; \qquad (\overline{\eta}_{\dot{\alpha}})^\dagger = \eta_\alpha. \end{aligned}$$

When multiplying four-vectors, one introduces a metric $\eta_{\mu\nu}$ such that $A_\mu = \eta_{\mu\nu} A^\nu$ in order to raise and lower indices. In Weyl spinor space, the metric is the 2D Levi-Civita $\epsilon_{\alpha\beta}$, i.e.

$$\epsilon_{\alpha\beta} = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which is antisymmetric, i.e.

$$\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma} \Rightarrow \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2 = -\epsilon_{\alpha\beta}.$$

As a result;

$$\begin{aligned} \xi_\alpha &= \epsilon_{\alpha\beta}\xi^\beta; \qquad & \overline{\eta}_{\dot{\alpha}} &= \epsilon_{\dot{\alpha}\dot{\beta}}\overline{\eta}^{\dot{\beta}}; \\ \xi^\alpha &= \epsilon^{\alpha\beta}\xi_\beta; \qquad & \overline{\eta}^{\dot{\alpha}} &= \epsilon^{\dot{\alpha}\dot{\beta}}\overline{\eta}_{\dot{\beta}}. \end{aligned}$$

These raising and lowering properties can be used to define the Lorentz-invariant spinor contractions, i.e.

$$\begin{aligned} \xi\eta &\equiv \xi^\alpha\eta_\alpha \\ &= \xi^\alpha\epsilon_{\alpha\beta}\eta^\eta && \text{Antisymmetric} \\ &= -\eta^\beta\epsilon_{\alpha\beta}\xi^\alpha && \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} \\ &= \eta^\beta\epsilon_{\beta\alpha}\xi^\alpha \\ &= \eta^\beta\xi_\beta \\ &\equiv \eta\xi. \end{aligned}$$

Similarly:

$$\begin{aligned} \overline{\xi}\overline{\eta} &\equiv (\eta\xi)^\dagger \\ &= (\eta^\alpha\xi_\alpha)^\dagger \\ &= \left(\eta^\alpha\epsilon_{\alpha\beta}\xi^\beta\right)^\dagger \\ &= \left(-\xi^\beta\epsilon_{\alpha\beta}\eta^\alpha\right)^\dagger \\ &= \left(\xi^\beta\epsilon_{\beta\alpha}\eta^\alpha\right)^\dagger \\ &= \left(\xi^\beta\eta_\beta\right)^\dagger \\ &\equiv \overline{\eta}\overline{\xi}. \end{aligned}$$

4.1.2 Dirac Spinors

The Dirac spinor is a four-dimensional complex vector comprised of two Weyl spinors as

$$\Psi_D(x) = \begin{pmatrix} \xi_\beta(x) \\ \overline{\eta}^{\dot{\beta}}(x) \end{pmatrix}; \qquad \overline{\Psi}_D(x) = \begin{pmatrix} \eta^\alpha(x) \\ \overline{\xi}_{\dot{\alpha}}(x) \end{pmatrix}^T,$$

where ξ are left-handed spinors $(\frac{1}{2}, 0)$ and η are right-handed spinors $(0, \frac{1}{2})$. The spinorial representation of this is then $\mathrm{SO}(1,3) \approx \frac{\mathrm{SL}(2,\mathbb{C})}{\mathbb{Z}_2}$.

The Dirac Equation

The Dirac equation is

$$\left(i\gamma^\mu \partial_\mu - m\right)\Psi = 0,$$

where

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot{\beta}} \\ (\overline{\sigma}^\mu)^{\dot{\alpha}\beta} & 0 \end{pmatrix},$$

with the Pauli matrix four-vector being given by $\sigma^\mu = (\mathbb{I}_2, \boldsymbol{\sigma})$; $\overline{\sigma}^\mu = (\mathbb{I}_2, -\boldsymbol{\sigma})$. The Dirac equation is analogous to the Klein-Gordon equation for spinors and is hence solved by the Dirac spinors.

Clifford Algebra

Clifford algebra is a unifying language for mathematics, and a reveleaing language for physics. Unlike the standard vector analysis, whose primitives are scalars and vectors for representing points and lines, Clifford algebra has additional spatial primitives for representing and volume segments in two and three dimensions. The spinor representation will be constructed in Clifford algebra by defining the following relations:

$$\begin{aligned} \{\sigma^\mu, \overline{\sigma}^\nu\} &\coloneqq \sigma^\mu \overline{\sigma}^\nu + \overline{\sigma}^\nu \sigma^\mu = 2\eta^{\mu\nu}\mathbb{I}_2 \\ \{\overline{\sigma}^\mu, \sigma^\nu\} &\coloneqq \overline{\sigma}^\mu \sigma^\nu + \sigma^\nu \overline{\sigma}^\mu = 2\eta^{\mu\nu}\mathbb{I}_2 \\ \{\gamma^\mu, \gamma^\nu\} &\coloneqq \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}\mathbb{I}_4. \end{aligned}$$

The Lagrangian for Dirac Fermions

Given that Dirac fermions are given by

$$\Psi_D(x) = \begin{pmatrix} \xi_\beta(x) \\ \overline{\eta}^{\dot{\beta}}(x) \end{pmatrix}; \qquad \overline{\Psi}_D(x) = \begin{pmatrix} \eta^\alpha(x), & \overline{\xi}_{\dot{\alpha}}(x) \end{pmatrix},$$

the Lagrangian for the Dirac equation is then

$$\begin{aligned} \mathcal{L}_D &= \overline{\Psi}\left(i\gamma^\mu\partial_\mu - m\mathbb{I}_4\right)\Psi \\ &= \begin{pmatrix} \eta^\alpha(x), & \overline{\xi}_{\dot{\alpha}}(x) \end{pmatrix} \left[i \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot{\beta}} \\ (\overline{\sigma}^\mu)^{\dot{\alpha}\beta} & 0 \end{pmatrix} \partial_\mu - m \begin{pmatrix} \delta^\beta_\alpha & 0 \\ 0 & \delta^{\dot{\alpha}}_{\dot{\beta}} \end{pmatrix} \right] \begin{pmatrix} \xi_\beta(x) \\ \overline{\eta}^{\dot{\beta}}(x) \end{pmatrix} \\ &= \eta^\alpha i\left(\sigma^\mu\right)_{\alpha\dot{\beta}} \left(\partial_\mu \overline{\eta}^{\dot{\beta}}\right) + \overline{\xi}_{\dot{\alpha}} i\left(\overline{\sigma}^\mu\right)^{\dot{\alpha}\beta} \left(\partial_\mu \xi_\beta\right) - m\left(\eta^\alpha\xi_\alpha + \overline{\xi}_{\dot{\alpha}}\overline{\eta}^{\dot{\alpha}}\right), \end{aligned}$$

which is invariant under Lorentz transformation.

Charge Conjugation

The charge conjugation operator $\hat{C}$ is given by

$$\hat{C} \coloneqq i\gamma^0\gamma^2 = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}.$$

The charge conjugation of a Dirac spinor is hence

$$\begin{aligned} \Psi^C = \hat{C}\overline{\Psi}^T &= \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \begin{pmatrix} \eta^\alpha \\ \overline{\xi}_{\dot{\alpha}} \end{pmatrix} \\ &= \begin{pmatrix} \epsilon_{\alpha\beta}\eta^\alpha \\ \epsilon^{\dot{\alpha}\dot{\beta}}\overline{\xi}_{\dot{\alpha}} \end{pmatrix} \\ &= \begin{pmatrix} \eta_\beta \\ \overline{\xi}^{\dot{\beta}} \end{pmatrix}. \end{aligned}$$

4.1.3 Majorana Spinors

The Majorana spinor is defined by a spinor that is equal to its charge conjugate, i.e.

$$\Psi_M = \Psi_M^C.$$

An example of a Majorana spinor in terms of Weyl spinors is

$$\Psi_M = \begin{pmatrix} \xi_\alpha \\ \overline{\xi}^{\dot{\alpha}} \end{pmatrix}.$$

The Majorana equation is a relativistic wave equation similar to the Dirac equation, but includes the charge conjugate of a spinor:

$$\begin{aligned} -i\gamma^\mu \partial_\mu \Psi + m\Psi^C &= 0 \\ i\gamma^\mu \partial_\mu \Psi^C + m\Psi &= 0, \end{aligned}$$

where these equations are equivalent.

4.1.4 Supplementary Mathematics

A point in the space-time manifold is denoted with a four-position $x^\mu = \left(x^0, x^1, x^2, x^3\right)$, where $x^0 = t$ and $x^{1,2,3}$ are the spatial components. The laws of relativity are invariant under the Lorentz group, which is all Lorentz transformations of Minkowski space-time. Transformations of this group are linear transformations acting on four-vectors as

$$x'^\mu = \Lambda^\mu_\nu x^\nu.$$

The metric tensor $\eta = \text{diag}(1, -1, -1, -1)$ is used to raise ($\eta^{\mu\nu}$) and lower ($\eta_{\mu\nu}$) indices. The quadratic form of the four-position is given by

$$\begin{aligned} x^2 &= x^\mu x_\mu \\ &= x^\mu \eta_{\mu\nu} x^\nu \\ &= x^0 x^0 - x^1 x^1 - x^2 x^2 - x^3 x^3 \\ &= \left(x^0\right)^2 - (\mathbf{x})^2. \end{aligned}$$

By construction, this is Lorentz-invariant, thus

$$\begin{aligned} x'^2 &= x'^\mu x'_\mu \\ &= x'^\mu \eta_{\mu\nu} x'^\nu \\ &= \left(\Lambda^\mu_\rho x^\rho\right) \eta_{\mu\nu} \left(\Lambda^\nu_\tau x^\tau\right) \\ &\stackrel{!}{=} x^\rho \eta_{\rho\tau} x^\tau \text{ for invariance} \\ \Rightarrow \eta_{\rho\tau} &= \Lambda^\mu_\rho \eta_{\mu\nu} \Lambda^\nu_\tau. \end{aligned}$$

One can propose two constraints for the Lorentz transform:

1. $\det(\Lambda) = \pm 1$
2. $\Lambda^0_0 \geq 1$,

which defines the four disconnected pieces in parameter space.

Proof 4.1: Lorentz constraints

The determinant of a product of matrices is the product of the individual matrix determinants, i.e. $\det(AB) = \det(A)\det(B)$. Therefore, taking the determinant of $(1,3)$ is

$$\begin{aligned} \det(\Lambda\Lambda) &= \det(\Lambda)^2 = 1 \\ &\Rightarrow \det(\Lambda) = \pm 1. \end{aligned}$$

From the Lorentz-invariance of four-vectors:

$$\begin{aligned} \eta_{00} &= \Lambda^\mu_0 \eta_{\mu\nu} \Lambda^\nu_0 \\ &= \underbrace{\Lambda^0_0 \eta_{00} \Lambda^0_0}_{\text{temporal}} + \underbrace{\Lambda^0_i \eta_{ii} \Lambda^i_0}_{\text{spatial}} \\ &= (+1)\left(\Lambda^0_0\right)^2 + (-1)\left(\Lambda^i_0\right)^2 \\ &= \left(\Lambda^0_0\right)^2 - \left(\Lambda^i_0\right)^2 \stackrel{!}{=} 1 \text{ by definition of } \eta_{\mu\nu}. \end{aligned}$$

Therefore

$$\begin{aligned} \left(\Lambda^0_0\right)^2 &= 1 + \left(\Lambda^i_0\right)^2 \\ \Rightarrow \left(\Lambda^0_0\right)^2 &\geq 1 \text{ as } \left(\Lambda^i_0\right)^2 \geq 0. \end{aligned}$$

The second constraint distinguishes **orthochronous** (preserving direction of time) Lorentz transformations with $\Lambda^0_0 \geq \pm 1$ and **non-orthochronous** Lorentz transforms with $\Lambda^0_0 \leq \pm 1$. The matrices Λ^μ_ν satisfying $(1,3)$, i.e. satisfying $\eta_{\rho\tau} = \Lambda^\mu_\rho \eta_{\rho\tau} \Lambda^\nu_\tau$, form a non-compact Lie group called the Lorentz group, L;

$$L \coloneqq \mathcal{O}(1,3) = \left\{ \Lambda \in \underbrace{\mathrm{GL}\,(4,\mathbb{R})}_{\substack{\text{invertible} \\ 4\times 4 \text{ real} \\ \text{matrices}}} | \Lambda^T \eta \Lambda = \eta \right\},$$

with Lie algebra

$$\sigma(1,3) \coloneqq \left\{ \sigma \in \underbrace{\mathrm{M}\,(4,\mathbb{R})}_{\substack{4\times 4 \text{ real} \\ \text{matrices}}} | \sigma^T = -\eta\sigma\eta \right\}.$$

Proof 4.2

From Lie algebra theory, it is known that each $\Lambda \in \mathcal{O}(1,3)$ can be written

$$\Lambda(t) = \exp(\sigma t),$$

where t is a real parameter and $\sigma \in \mathcal{O}(1,3)$ is an element of the Lie algebra. Matrices of $\mathcal{O}(1,3)$ are subject to the condition

$$\Lambda^T(t)\eta\Lambda(t) = \eta.$$

Combining the two above expressions, gives

$$\big(\exp(\sigma t)\big)^T \eta \exp(\sigma t) = \eta.$$

The time-derivative of this is then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\big(\exp(\sigma t)\big)^T \eta \exp(\sigma t)\right)\Big|_{t=0} = 0,$$

thus one obtains the condition for Lie algebra elements, since the Lie algebra for any Lie group is **isomorphic** (...) to the tangent space at

the identity of the group. It then follows that

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\exp(\sigma t)\right)^T\right)\eta\exp(\sigma t)+\left(\exp(\sigma t)\right)^T\eta\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp(\sigma t)\right)\right]_{t=0}=0$$
$$\therefore \sigma^T\eta+\eta\sigma=0$$
$$\Rightarrow \sigma^T=-\eta\sigma\eta$$

from $\sigma^T\eta\eta \ \ \forall\sigma\in\mathcal{O}(1,3)$ as $\eta\eta=1$.

In summary, the classification of Λ being any invertible 4×4 real matrix with real elements, i.e. $\Lambda\in\mathrm{GL}(4,\mathbb{R})$, has been used to derive properties of the Lorentz group.

4.2 Quantisation of the Dirac Fermion Field

One is now able to quantise the theory of the Dirac Lagrangian

$$\mathcal{L}=\overline{\Psi}\left(i\partial\!\!\!/-m\right)\Psi,$$

where the slashed partial is defined as $\partial\!\!\!/\coloneqq\gamma^\mu\partial_\mu$.

The theory at hand is a free scalar theory, in which any classical solutions are sums of plane waves, and hence one can write the quantum operators as

$$\Psi(x)\mapsto\hat{\Psi}(x)=\sum_{s=\pm\frac{1}{2}}\int_{\mathbf{k}}\left(b\left(\mathbf{k},s\right)u\left(\mathbf{k},s\right)e^{-ik\cdot x}+d^\dagger\left(\mathbf{k},s\right)v\left(\mathbf{k},s\right)e^{ik\cdot x}\right)$$
$$\Psi^\dagger(x)\mapsto\overline{\Psi}(x)=\sum_{s=\pm\frac{1}{2}}\int_{\mathbf{k}}\left(b^\dagger\left(\mathbf{k},s\right)u^\dagger\left(\mathbf{k},s\right)\gamma^0e^{ik\cdot x}+d\left(\mathbf{k},s\right)v^\dagger\left(\mathbf{k},s\right)\gamma^0e^{-ik\cdot x}\right)$$

where $k^2=k^\mu k_\mu=m^2\Leftrightarrow k^0=E_{\mathbf{k}}=\sqrt{\mathbf{k}^2+m^2}$. The quantum operators have 8 degrees of freedom (4 dimensions, each with a spin of $\pm\frac{1}{2}$), however there are only 4 entries. This is due to constraints placed by the Dirac equation, however this is not true for the Klein-Gordon equation. The operators $b^\dagger\left(\mathbf{k},s\right)$ create particles associated to the spinors $u\left(\mathbf{k},s\right)$, whilst

$d^{\dagger}(\mathbf{k}, s)$ create particles associated to $v(\mathbf{k}, s)$. For now, consider only simple fermions, i.e. electrons and positrons. The creation operators then act upon the vacuum as

$$b^{\dagger}(\mathbf{k}, s)|0\rangle = \left|e^{-}(\mathbf{k}, s)\right\rangle$$
$$d^{\dagger}(\mathbf{k}, s)|0\rangle = \left|e^{+}(\mathbf{k}, s)\right\rangle.$$

The spinors u and v are defined as the solutions of

$$\left(\not{p} - m\right) u(\mathbf{p}, s) = 0$$
$$\left(\not{p} + m\right) v(\mathbf{p}, s) = 0.$$

The general solutions to these are

$$u(\mathbf{k}, s) = \frac{\not{k} + m}{\sqrt{2m(E_k + m)}} u(0, s);$$
$$v(\mathbf{k}, s) = \frac{-\not{k} + m}{\sqrt{2m(E_k + m)}} v(0, s),$$

where $u(0, s)$ and $v(0, s)$ are the spinors in their rest frames. These spinors satisfy normalisation properties, which make use of the relations $\overline{u} \coloneqq u^{\dagger}\gamma^0$ and $\overline{v} \coloneqq v^{\dagger}\gamma^0$ with $u = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I}_2 & -\mathbb{I}_2 \\ \mathbb{I}_2 & \mathbb{I}_2 \end{pmatrix}$;

$$\overline{u}(\mathbf{k}, s)\, u\left(\mathbf{k}, s'\right) = 2m\delta_{ss'};$$
$$\overline{v}(\mathbf{k}, s)\, v\left(\mathbf{k}, s'\right) = -2m\delta_{ss'}.$$

The Dirac and Weyl quantum operators are related by

$$\Psi_D = u^{\dagger}\Psi_W$$
$$\gamma^{\mu}_D = u^{\dagger}\gamma^{\mu}_W u.$$

4.2.1 Conjugate Momentum

One defines the momentum in the usual manner, by considering the Lagrangian $\mathcal{L} = \overline{\Psi}\left(i\partial\!\!\!/ - m\right)\Psi$;

$$\begin{aligned}\Pi &\coloneqq -\frac{\partial\mathcal{L}}{\partial(\partial_0\Psi)}\\ &= i\overline{\Psi}\gamma^0\\ &\equiv i\Psi^\dagger,\end{aligned}$$

where the initial negative sign derives from the Grassmann property of these numbers. For the Dirac Lagrangian, the momentum conjugate to Ψ is $i\Psi^\dagger$. It does not involve the time derivative of Ψ. This is as it should be for an equation of motion that is first order in time, rather than second.

The new quantum operators satisfy the canonical equal-time anti-commutation relations are

$$\begin{aligned}\left\{\Psi_\alpha(\mathbf{x}), \Psi_\beta(\mathbf{y})\right\} &= \left\{\overline{\Psi}_\alpha(\mathbf{x}), \overline{\Psi}_\beta(\mathbf{y})\right\} = 0;\\ \left\{\Psi_\alpha(\mathbf{x}), \overline{\Psi}_\beta(\mathbf{y})\right\} &= i\delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y}).\end{aligned}$$

These are equivalent to

$$\left\{b(\mathbf{k},s), b^\dagger\left(\mathbf{k}',s'\right)\right\} = \left\{d(\mathbf{k},s), d^\dagger\left(\mathbf{k}',s'\right)\right\} = (2\pi)^3 2E_{\mathbf{k}}\delta_{ss'}\delta^{(3)}\left(\mathbf{k}-\mathbf{k}'\right),$$

with all other commutations vanishing.

4.2.2 The Fermion Propagator

The time-ordered product for fermions is given by

$$T\left\{\Psi(x)\overline{\Psi}\left(x'\right)\right\} = \Theta\left(t-t'\right)\Psi(x)\overline{\Psi}\left(x'\right) - \Theta\left(t-t'\right)\overline{\Psi}\left(x'\right)\Psi(x),$$

where the negative sign for the right-most term derives from fermion fields being Grassmann in nature. Performing a similar calculation as was carried

out with scalar fields, one gets the definition of the **fermion propagator** S_F

$$\left\langle 0 \left| T\left\{\Psi\left(x\right)\overline{\Psi}\left(x'\right)\right\} \right| 0 \right\rangle = \int \frac{\mathrm{d}^4k}{\left(2\pi\right)^4} \frac{i\exp\left(-ik\left(x-x'\right)\right)}{\not{k}-m+i\epsilon} \coloneqq iS_F\left(x-x'\right),$$

which is often written alternatively as

$$\begin{aligned} iS_F\left(x-x'\right) &= \int \frac{\mathrm{d}^4k}{\left(2\pi\right)^4} \frac{i\left(\not{k}+m\right)\exp\left(-ik\left(x-x'\right)\right)}{k^2-m^2+i\epsilon} \\ &= \left(i\not{\partial}_x+m\right) i\Delta_F\left(x-x'\right), \end{aligned}$$

where $\Delta_F\left(x-x'\right)$ is the Feynman scalar propagator. This is because

$$\begin{aligned} \frac{1}{\not{k}-m} &= \frac{\not{k}+m}{\left(\not{k}+m\right)\left(\not{k}-m\right)} \\ &= \frac{\not{k}+m}{\not{k}^2-m^2}, \end{aligned}$$

where $\not{k}^2 = k^\mu k^\nu \gamma_\mu\gamma_\nu = k^\mu k^\nu\left(\frac{1}{2}\left\{\gamma_\mu,\gamma_\nu\right\}\right) = k^\mu k^\nu \eta_{\mu\nu}\mathbb{I}_4 = k^\mu k_\mu \mathbb{I}_4 = k^2\mathbb{I}_4$. The Feynman rules for such a propagator is as follows:

Element	**Diagram**	**Transition Amplitude,** iT_{fi}
Fermion propagator	$\overline{\Psi}_\alpha$ (x′) —k→ Ψ_β (x)	$\left(\frac{i}{\not{k}-m+i\epsilon}\right)_{\alpha\beta} = \frac{i\left(\not{k}+m\right)}{k^2-m^2+i\epsilon}$

4.3 Gauge Symmetry

Noether's theorem will be reconsidered, instead using the Dirac Lagrangian

$$\mathcal{L}_D = \overline{\Psi}\left(i\not{\partial}-m\right)\Psi.$$

4.3.1 Global U(1) Transformation

Recall that a global $U(1)$ transformation is a rotation by some phase θ that is independent of position, i.e.

$$\begin{aligned}\Psi &\mapsto \Psi' = e^{i\theta}\Psi;\\ \overline{\Psi} &\mapsto \overline{\Psi}' = e^{-i\theta}\overline{\Psi}.\end{aligned}$$

As a result, the transformed Dirac Lagrangian becomes

$$\begin{aligned}\mathcal{L}'_D &= \overline{\Psi}' \left(i\partial\!\!\!/ - m\right) \Psi'\\ &= e^{-i\theta}\overline{\Psi} \left(i\partial\!\!\!/ - m\right) e^{i\theta}\Psi.\end{aligned}$$

As θ is independent of any variables, the exponents can be moved such that they cancel one another, and hence the Dirac Lagrangian is invariant under global $U(1)$ transformation, i.e.

$$\mathcal{L}_D \left[\overline{\Psi}', \Psi'\right] = \mathcal{L}_D \left[\overline{\Psi}, \Psi\right].$$

4.3.2 Local U(1) Transformation

Under a local $U(1)$ transformation, the phase θ has dependence on space-time location, i.e. $\theta = \theta(x)$, where x is the four-position describing the transformation's location. The transformed wavefunctions are given by

$$\begin{aligned}\Psi &\mapsto \Psi' = e^{i\theta(x)}\Psi;\\ \overline{\Psi} &\mapsto \overline{\Psi}' = e^{-i\theta(x)}\overline{\Psi}.\end{aligned}$$

The transformed Lagrangian is hence

$$\begin{aligned}\mathcal{L}'_D &= \overline{\Psi}' \left(i\partial\!\!\!/ - m\right) \Psi'\\ &= e^{-i\theta(x)}\overline{\Psi} \left(i\partial\!\!\!/ - m\right) e^{i\theta(x)}\Psi\\ &= e^{-i\theta(x)}\overline{\Psi} \left(i\partial\!\!\!/ \left(e^{i\theta(x)}\right)\Psi + ie^{i\theta(x)}\partial\!\!\!/ \left(\Psi\right) - me^{i\theta(x)}\Psi\right)\\ &= e^{-i\theta(x)}\overline{\Psi} \left(i\left(i\partial\!\!\!/ \left(\theta\right)\right)e^{i\theta(x)}\Psi + ie^{i\theta(x)}\partial\!\!\!/ \left(\Psi\right) - me^{i\theta(x)}\Psi\right).\end{aligned}$$

At this point the exponents can be brought to cancel out, thus the transformed Lagrangian is

$$\begin{aligned}\mathcal{L}'_D &= \underbrace{\overline{\Psi}\left(i\partial\!\!\!/ - m\right)\Psi}_{=\mathcal{L}_D} - \underbrace{\overline{\Psi}\left(\partial\!\!\!/\left(\theta\right)\right)\Psi}_{\coloneqq\delta\mathcal{L}_D=\overline{\Psi}\gamma^\mu\left(\partial_\mu\theta\right)\Psi} \\ &= \mathcal{L}_D + \delta\mathcal{L}_D.\end{aligned}$$

One wishes to have invariance under the transformation, thus one can introduce a vector four-potential A_μ to compensate, i.e. a photon. The new Lagrangian term is then

$$\mathcal{L}_{A\overline{\Psi}\Psi} = -eA_\mu\overline{\Psi}\gamma^\mu\Psi.$$

Under a local $U(1)$ transformation the four-potential becomes

$$\begin{aligned}A_\mu \overset{U(1)}{\longmapsto} A'_\mu &= A_\mu - \frac{1}{e}\partial_\mu\left(\theta\left(x\right)\right) && \text{by definition} \\ \Rightarrow \mathcal{L}_{A\overline{\Psi}\Psi} \overset{U(1)}{\longmapsto} \mathcal{L}'_{A\overline{\Psi}\Psi} &= \mathcal{L}_{A\overline{\Psi}\Psi} + \overline{\Psi}\gamma^\mu\left(\partial_\mu\theta\right)\Psi.\end{aligned}$$

One can immediately see that the second term exactly cancels $\delta\mathcal{L}$ from earlier, hence the total Lagrangian that is invariant under $U(1)$ transformation is

$$\begin{aligned}\mathcal{L}_\Psi &= \mathcal{L}_D + \mathcal{L}_{A\overline{\Psi}\Psi} \\ &= \overline{\Psi}\left(i\partial\!\!\!/ - eA\!\!\!/ - m\right)\Psi.\end{aligned}$$

This inclusion of Lagrangians to satisfy invariance under given transformations is known as **gauge invariance**. Gauge symmetry gives rise to a conserved electric current and charge according to Noether's theorem. However, the gauge symmetry demonstrated here does not imply that a massless photon is the only solution, nor does it prove that the photon exists.

4.4 Quantisation of the Electromagnetic Field

The electromagnetic Lagrangian is related to the electromagnetic field tensor by

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The quantum electrodynamical Lagrangian is then

$$\begin{aligned}\mathcal{L}_{\text{QED}} &= \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{D}} + \mathcal{L}_{A\overline{\Psi}\Psi} \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\Psi}\left(i\partial\!\!\!/ - eA\!\!\!/ - m\right)\Psi.\end{aligned}$$

One may now ask how to quantise A_μ, and in so doing one derives quantum electrodynamics. Consider a vector potential given by

$$A_\mu(x) = \sum_{\lambda=0}^{3}\int_{\mathbf{k}}\left(a(\mathbf{k},\lambda)\,\epsilon_\mu(\mathbf{k},\lambda)\,e^{-ik\cdot x} + a^\dagger(\mathbf{k},\lambda)\,\epsilon_\mu^*(\mathbf{k},\lambda)\,e^{ik\cdot x}\right)$$

with conjugate momentum $\Pi_\mu \coloneqq \partial_t A_\mu$. The vector potential carries 4 real degrees of freedom corresponding to $\lambda = 0, 1, 2, 3$, where $\lambda = 0$ is for timelike photons, $\lambda = 1, 2$ are transverse and $\lambda = 3$ are longitudinal. Only the transverse photons are real. The photons are described by polarisation vectors $\epsilon_\mu(\mathbf{k}, \lambda)$. The vector four-potential and its conjugate momentum satisfy the equal-time commutation relation

$$\left[A_\mu(t,\mathbf{x}), \Pi_\nu(t,\mathbf{y})\right] = -i\eta_{\mu\nu}\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$

with all other commutations vanishing.

4.4.1 Properties of Polarisation Vectors

There are now 4 polarisation four-vectors ϵ_μ, from which one chooses ϵ_0 to be time-like and $\epsilon_{1,2,3}$ to be space-like. By convention the normalisation of these is

$$\epsilon_\mu \cdot \epsilon_\nu = \eta_{\mu\nu},$$

which also means

$$\left(\epsilon_\mu\right)^\lambda \left(\epsilon_\nu\right)^{\lambda'} \eta_{\lambda\lambda'} = \eta_{\mu\nu}.$$

The polarisation vectors depend on the photon four-momentum $p = \left(|\mathbf{p}|, \mathbf{p}\right)$ of the space-like polarisations. $\lambda = 1, 2$ will be chosen to be transverse to the momentum, i.e.

$$\epsilon_1 \cdot p = \epsilon_2 \cdot p = 0.$$

The third spacelike polarisation vector ϵ_3 is the longitudinal polarisation. If one assumes a fictitious photon mass $m_\gamma \to 0$, one possible solution is

$$\begin{aligned}
\epsilon_\mu(\mathbf{k},0) &= \frac{1}{m_\gamma} k_\mu; \\
\epsilon_\mu(\mathbf{k},1) &= \frac{1}{\sqrt{2}}(0,1,i,0); \\
\epsilon_\mu(\mathbf{k},2) &= \frac{1}{\sqrt{2}}(0,1,-i,0); \\
\epsilon_\mu(\mathbf{k},3) &= \frac{1}{m_\gamma}\left(|\mathbf{k}|, k^0 \frac{\mathbf{k}}{|\mathbf{k}|}\right).
\end{aligned}$$

Physical observables, such as S-matrix elements, do not have singularities in the limit $m_\gamma \to 0$. Further properties of the polarisation vectors include:

$$\begin{aligned}
k^\mu \epsilon_\mu(\mathbf{k}, \lambda = 1,2,3) &= 0; \\
\epsilon_\mu(\mathbf{k},\lambda)\,\epsilon^{\mu^*}\left(\mathbf{k},\lambda'\right) &= -(-1)^{\delta_{\lambda,0}}\,\delta_{\lambda\lambda'}; \\
\left[a_{\mathbf{k},\lambda}, a^\dagger_{\mathbf{k}',\lambda'}\right] &= (-1)^{\delta_{\lambda,0}}\,\delta_{\lambda\lambda'} 2E_{\mathbf{k}}\delta^{(3)}\left(\mathbf{k}-\mathbf{k}'\right)(2\pi)^3; \\
\sum_{\lambda=0}^{3} (-1)^{\delta_{\lambda,0}}\,\epsilon_\mu(\mathbf{k},\lambda)\,\epsilon^*_\nu(\mathbf{k},\lambda) &= -\eta_{\mu\nu}.
\end{aligned}$$

4.5 The Photon Propagator and Gauge Fixing

In QED, the TOP of vector field four-potentials is

$$\begin{aligned}
\left\langle 0 \left| T\left\{A_\mu(x) A_\nu(y)\right\} \right| 0 \right\rangle &= i\Delta_{\mu\nu}(x-y) \\
&= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{k^2 + i\epsilon} e^{-ik\cdot(x-y)},
\end{aligned}$$

where $T\left\{A_\mu(x)A_\nu(y)\right\} \coloneqq \Theta\left(x^0-y^0\right) A_\mu(x)A_\nu(y) + \Theta\left(y^0-x^0\right) A_\nu(y)A_\mu(x)$.

Alternatively, one may add a gauge-fixing term

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2$$

to the QED Lagrangian

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\Psi}\left(i\partial\!\!\!/ - eA\!\!\!/ - m\right)\Psi$$

in order to break the gauge symmetry, which is necessary to remove unphysical degrees of freedom of the photon $A_\mu(x)$.

One should note that the new term is not invariant, hence fixing the gauge. This term disappears in the limit $\xi \to \infty$ and is called the **unitary gauge**. The Euler-Lagrange equation for the photon is then

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial\mathcal{L}}{\partial A_\nu} = 0,$$

which gives

$$\left(\eta_{\mu\nu}\underbrace{\partial_k\partial^k}_{\equiv\Box} - \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial_\nu\right)A^\nu(x) = 0.$$

Introducing a photon propagator $\Delta^{(\xi)}_{\mu\nu}$ to act as a Green's function in solving the above, i.e. $\left(\eta_{\mu\nu}\underbrace{\partial_k\partial^k}_{\equiv\Box} - \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial_\nu\right)\Delta^{(\xi)}_{\mu\nu} = \delta^\mu_\nu\delta^{(4)}(x-y)$, then

$$\Delta^{(\xi)}_{\mu\nu}(x-y) = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\left(-\eta_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\frac{e^{-ik\cdot(x-y)}}{k^2 + i\epsilon}.$$

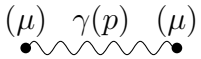
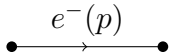
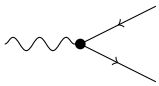
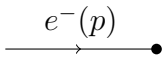
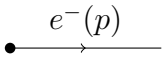
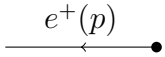
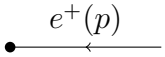
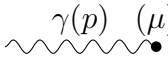
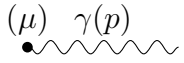
One may then see that the photon propagator derived through canonical quantisation is the same as the Green's function photon propagator except that $\xi = 1$, which is the **Feynman-'t Hooft gauge**. The S-matrix elements are independent of ξ. Using the QED Lagrangian, one can derive the following Feynman rules:

Example 4.2: Using Feynman Rules

Consider the scattering interaction

$$e^-\left(p_1, s_e\right)\mu^+\left(p_2, s_\mu\right) \longrightarrow e^-\left(k_1, s'_e\right)\mu^+\left(k_2, s'_\mu\right),$$

as shown in Figure 4.2.

Diagram	Transition Amplitude
	$\frac{-i\eta_{\mu\nu}}{p^2+i\epsilon}$
	$\frac{i}{\not{p}-m+i\epsilon}$
	$ie\gamma_\mu$
	$u(p)$ for e^- in initial state
	$\overline{u}(p)$ for e^- in final state
	$\overline{v}(p)$ for e^+ in initial state
	$v(p)$ for e^+ in final state
	$\epsilon^\mu(\mathbf{p},\lambda)$ for γ in initial state
	$\epsilon^{\mu^*}(\mathbf{p},\lambda)$ for γ in final state.

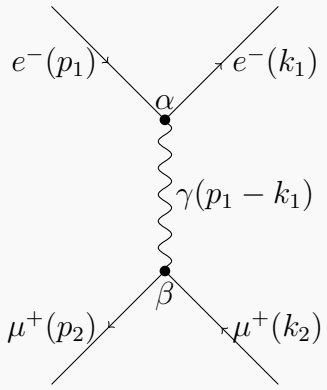
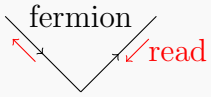


Figure 4.2: asd

To calculate the matrix element, read *against* fermionic arrows, i.e.



The matrix elements are hence

Electron: $\overline{u}_e\left(k_1, s_e'\right)\left(-ie\gamma_\alpha\right) u_e\left(p_1, s_e\right)$

Photon: $-\frac{i\eta^{\alpha\beta}}{(p_1-k_1)^2}$

Muon: $\overline{v}_\mu\left(p_2, s_\mu\right)\left(-ie\gamma_\beta\right) v_\mu\left(k_2, s_\mu'\right)$.

The matrix element is then the individual components multiplied together;

$$\begin{aligned}\mathcal{M}_{fi} &= \left(\overline{u}_e\left(k_1, s_e'\right)\left(-ie\gamma_\alpha\right) u_e\left(p_1, s_e\right)\right) \cdot \left(-\frac{i\eta^{\alpha\beta}}{\left(p_1 - k_1\right)^2}\right) \\ &\quad \cdot \left(\overline{v}_\mu\left(p_2, s_\mu\right)\left(-ie\gamma_\beta\right) v_\mu\left(k_2, s_\mu'\right)\right) \\ &= \frac{ie^2}{t}\left(\overline{u}_e'\gamma_\alpha u_e\right)\left(\overline{v}_\mu\gamma_\beta v_\mu'\right),\end{aligned}$$

where $\left(p_1 - k_1\right)^2 = t$ has been used. Using previously mentioned/derived relations ($\overline{u} = u^\dagger\gamma^0; \left(\gamma^0\right)^2 = \mathbb{I}_4; \overline{\gamma}^\mu = \gamma^\mu; \mathrm{Tr}(AB) =$

$\mathrm{Tr}(BA); \overline{AB} = \overline{B} \cdot \overline{A}$), the expected matrix element amplitude is hence

$$\begin{aligned}\left\langle \left|\mathcal{M}_{fi}\right|^2 \right\rangle &= \frac{1}{4} \sum_{s_{e,\mu}, s'_{e,\mu}} \overline{u}'_e \gamma_\alpha u_e \underbrace{\overline{\overline{u}'_e \gamma_\beta u_e}}_{=\overline{u}_e \gamma_\beta u'_e} \overline{v}_\mu \gamma^\alpha v'_\mu \underbrace{\overline{\overline{v}_\mu \gamma^\beta v'_\mu}}_{=\overline{v}'_\mu \gamma^\beta v_\mu} \\ &= \frac{1}{4} \mathrm{Tr}\left(\gamma_\alpha \left(\not{p}_1 + m_e \right) \gamma_\beta \left(\not{k}_1 + m_e \right) \right) \mathrm{Tr}\left(\ldots \right).\end{aligned}$$

Sidenotes on Clifford Algebra

$$\mathrm{Tr}\left(\gamma^\mu \gamma^\nu\right) = \frac{1}{2} \mathrm{Tr}\left(\underbrace{\{\gamma^\mu, \gamma^\nu\}}_{=2\eta^{\mu\nu}\mathbb{I}_4} \right) = \eta^{\mu\nu} \mathrm{Tr}\left(\mathbb{I}_4\right) = 4\eta^{\mu\nu}.$$

$$\begin{aligned}\mathrm{Tr}\left(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma\right) &= 2\eta^{\mu\nu} \mathrm{Tr}\left(\gamma^\rho \gamma^\sigma\right) - \mathrm{Tr}\left(\gamma^\nu \gamma^\mu \gamma^\rho \gamma^\sigma\right) \\ &= 8\eta^{\mu\nu}\eta^{\rho\sigma} - 2\eta^{\mu\rho} \mathrm{Tr}\left(\gamma^\nu \gamma^\sigma\right) + \mathrm{Tr}\left(\gamma^\nu \gamma^\rho \gamma^\mu \gamma^\sigma\right) \\ &= 8\eta^{\mu\nu}\eta^{\rho\sigma} - 8\eta^{\mu\rho}\eta^{\nu\sigma} + 8\eta^{\nu\rho}\eta^{\mu\sigma} - \underbrace{\mathrm{Tr}\left(\gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu\right)}_{\substack{\text{cyclically} \\ \text{identical to LHS}}} \\ \Rightarrow \mathrm{Tr}\left(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma\right) &= 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\nu\rho}\eta^{\mu\sigma}\right).\end{aligned}$$

4.6 Becchi-Rouet-Stora Transformations

4.7 Feynman Rules for Quantum Electrodynamics

Chapter 5

Renormalisation

Definition 5.1: Renormalisation

The term **renormalisation** refers to the technique of removing infinities from field theories such that theoretical predictions can match experimental outcomes. Roughly speaking, one redefines unmeasurable quantities as to absorb divergences to make measurable quantities finite.

5.1 Renormalisability

5.1.1 Superficial Degree of Divergence (SDD)

To define and illustrate the concept of the superficial degree of freedom (SDD) of the diagram, it is useful to refer to the following scalar Lagrangian:

$$\mathcal{L} = \frac{1}{2} z \left(\partial_\mu \phi\right) \left(\partial^\mu \phi\right) - \frac{1}{2} z m^2 \phi^2 - \sum_{n=3}^{\infty} z_n \frac{g_n}{n!} \phi^n.$$

For a diagram has p propagators (internal lines) and L loops in d-dimensional space-time, there will be total powers of $2p$ loop-momenta in the denominator and dL loop-momenta in the numerator. The difference of these values D is

hence defined as

$$D \coloneqq dL - 2p.$$

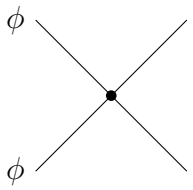
This is called the superficial degree of divergence, i.e. if $D \geq 0$ the diagram is superifically divergent.

5.1.2 ϕ^4-Theory

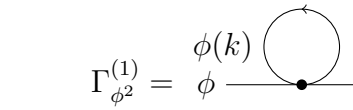
Consider a ϕ^4-theory whose Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m^2\phi^2 - \underbrace{\frac{\lambda}{4!}\phi^4}_{=\mathcal{L}_{\text{int}}}.$$

The tree-level diagram for this is

$\Gamma^{(0)}_{\phi^4} =$  $: -i\lambda.$

The ϕ^2 1-particle irreducible (1PI) loop graph is

$\Gamma^{(1)}_{\phi^2} =$  $: -\frac{i\lambda}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon}.$

In such a case, there is 1 loop and 1 propagator ($\phi(k)$), thus the SDD is $D_{\Gamma^{(1)}_{\phi^2}} = (4 \times 1) - (2 \times 1) = 2$. For large momenta, i.e. $k \gg m$, this diverges, known as the **ultraviolet (UV) limit**. In order to normalise this, a regularisation cutoff denoted Λ^D is introduced, where D is the SDD. The cutoff regularisation for the above loop diagram is given by

$$-\frac{i\lambda}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon} \xmapsto{|k|\gg m} \propto \int_0^{\Lambda}\frac{|k|^3\ \mathrm{d}k}{|k|^2} = \Lambda^2.$$

The effect of this cutoff is easiest seen in Euclidean space. Consider a Wick rotation on the four-momentum k, as shown in Figure 5.1. The integral over

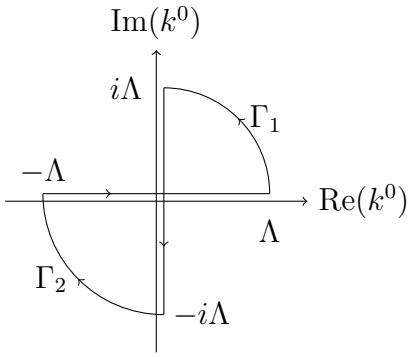


Figure 5.1

k^0 then gives

$$\oint \mathrm{d}k^0 = \int_{-\Lambda}^{\Lambda} \mathrm{d}k^0 + i\int_{\Lambda}^{-\Lambda} \mathrm{d}k_E^0 + \underbrace{\int_{\Gamma_1} \mathrm{d}k^0 + \int_{\Gamma_2} \mathrm{d}k^0}_{\text{cancel}},$$

where E denotes Euclidean space-time. The total integral must be equal to zero as there are no singularities, and so one concludes that

$$\int_{-\Lambda}^{\Lambda} \mathrm{d}k^0 = i\int_{-\Lambda}^{\Lambda} \mathrm{d}k_E^0.$$

As such, the following mappings occur:

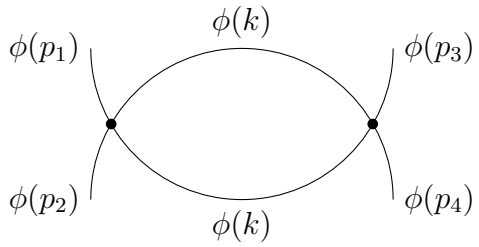
$$\begin{gathered}\int_{-\infty}^{\infty} \frac{\mathrm{d}k^0}{2\pi} \mapsto i\int_{-\infty}^{\infty} \frac{\mathrm{d}k_E^0}{2\pi};\\ k^0 \mapsto ik_E^0;\\ \left(k^0\right)^2 \mapsto -\left(k_E^0\right)^2;\\ \eta_{\mu\nu} \mapsto \delta_{\mu\nu}.\end{gathered}$$

The overall integral for the ϕ^2 1PI loop graph is hence

$$
\begin{aligned}
-\frac{i\lambda}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon} &\mapsto \left(-\frac{i\lambda}{2}\right) i^2\int_{-\infty}^{\infty}\frac{\mathrm{d}_E^4k}{(2\pi)^4}\frac{i}{-k_E^2-m^2+i\epsilon} \\
&= \left(-\frac{i\lambda}{2}\right)\int_0^{\Lambda}\frac{(2\pi)^2\overbrace{|k_E|^3\,\mathrm{d}k_E}^{=\frac{1}{2}|k_E|^2\ \mathrm{d}|k_E|^2}}{(2\pi)^4}\frac{1}{k_E^2+m^2} \\
&= \left(-\frac{i\lambda}{2}\right)\int_0^{\Lambda}\frac{x\ \mathrm{d}x}{2\,(2\pi)^2}\frac{1}{x+m^2} \qquad (x=|k_E|^2) \\
&= \left(-\frac{i\lambda}{32\pi^2}\right)\left(\Lambda^2-m^2\ln\left(\frac{\Lambda}{m^2}\right)\right).
\end{aligned}
$$

The ϕ^4 1PI loop graph is given by

$\Gamma^{(1)}_{\phi^4} =$



$$
: \underbrace{\frac{(-i\lambda)^2}{2}}_{\substack{2\text{ lines}\\ \text{crossing}}}\int_{-\infty}^{\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\underbrace{\frac{i^2}{\left(k^2-m^2+i\epsilon\right)^2}}_{2\text{ propagators}}.
$$

The SDD of this is

$$
D_{\Gamma^{(1)}_{\phi^4}} = (4\times 1)-(2\times 2) = 0,
$$

which is a logarithmic divergence, i.e.

$$
\frac{(-i\lambda)^2}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i^2}{\left(k^2-m^2+i\epsilon\right)^2} \overset{|k|\gg m}{\longmapsto} \int_0^{\Lambda}\frac{|k|^3\ \mathrm{d}k}{k^4} = \int_0^{\Lambda}\frac{\mathrm{d}k}{k} = \ln(\Lambda).
$$

5.2 Weinberg's Theorem of Renormalisability

Weinberg's theorem of renormalisability states that a 1PI loop is UV-finite if D_Γ is negative and the superficial degree of divergence of all possible subgraphs are also negative. Possible caveats to the theorem may result from

additional symmetries acting upon the theory, e.g. gauge symmetries, chiral symmetries, supersymmetry.

5.3 Renormalisability of ϕ^n-Theories

5.3.1 4 Dimensions

ϕ^3-Theory

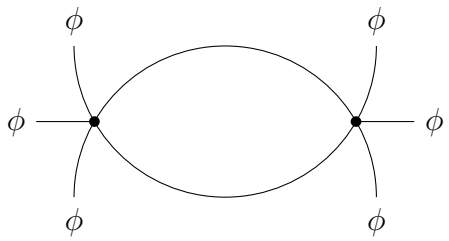
In ϕ^3-theory, there are a finite number of diagrams that are UV-finite. Such a theory is called super-renormalisable.

ϕ^4-Theory

New divergences occur at each loop order. However, all of these UV divergences are related to the 1PI amplitudes with a finite number of external legs (ϕ fields).

$\phi^{\geq 5}$-Theory

All $\Gamma_{\phi^{n>4}}$ amplitudes are UV divergent for a high enough order in perturbation theory. For example, the following figure shows Γ_{ϕ^6} and has $D = 0$:



There are three levels of renormalisability:

Super-renormalisable:	Only a finite number of Feynman diagrams are superficially divergent $\Rightarrow [\lambda] > 0$.
Renormalisable:	Only a finite number of amplitudes are superficially divergent, but divergences occurs in all orders in perturbation theory $\Rightarrow [\lambda] = 0$.
Non-renormalisable:	All amplitudes are divergent at a sufficiently high order in perturbation theory $\Rightarrow [\lambda] < 0$.

If a physical system satisfies classical equations of motion it is said to be on-(mass)-shell, otherwise it is off-shell. Virtual particles cannot satisfy the Einstein energy-momentum relationship and so are termed off-shell. The on-mass-shell renormalisation conditions are:

1. Polar propagator on-shell renormalisation: $\Gamma_{\phi^2}\left(p^2 = m_R^2\right) = 0$
2. Residual propagator on-shell renormalisation: $-i \left.\dfrac{\mathrm{d}\,\Gamma_{\phi^2}\left(p^2\right)}{\mathrm{d}p^2}\right|_{p^2=m_R^2} = 1$

Proof 5.1: Deriving normalisation parameters from on-shell renormalisation conditions

$\underline{\boldsymbol{\delta m^{2(1)}}}$

The loopwise expansion of ϕ^2 is

$$\begin{aligned}\Gamma_{\phi^2}\left(p^2\right) &= iz_\phi\left(p^2 - m_0^2\right) + (1\text{ loop}) + (2\text{ loop}) + \ldots \\ &= i\left(1 + \delta z_\phi^{(1)}\right)\left(p^2 - m_R^2 - \delta m^{2^{(1)}}\right) + \ldots,\end{aligned}$$

which for on-shell renormalisation is then

$$\begin{aligned}\Gamma_{\phi^2}\left(p^2 = m_R^2\right) &= i\left(1 + \delta z_\phi^{(1)}\right)\left(-\delta m^{2^{(1)}}\right) \\ \text{1 loop:} \qquad &+ \left(\frac{-i\lambda_0}{2}\right)\int_{-\infty}^{\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2 - m_0^2 + i\epsilon} \\ &+ \quad \vdots \\ &\overset{!}{=} 0 \text{ by condition 1.}\end{aligned}$$

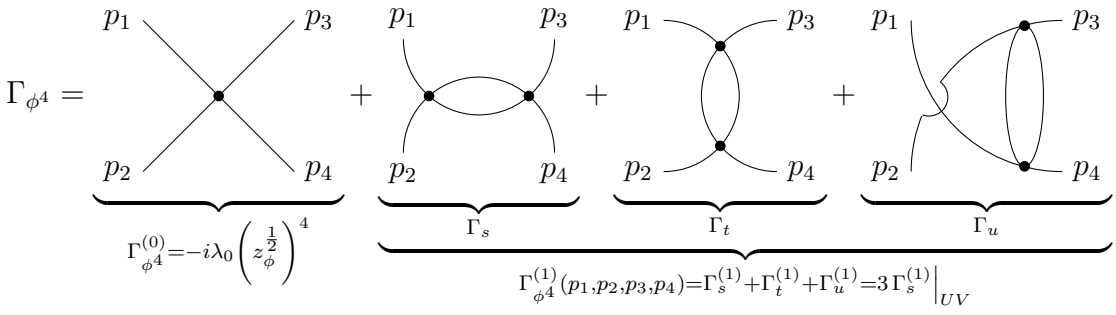
Therefore

$$
\begin{aligned}
i\delta m^{2^{(1)}} &= \frac{\lambda_R}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m_R^2} \\
&= -\frac{i\lambda_R}{32\pi^2}\left(\Lambda^2 - m_R^2\ln\left(\frac{\Lambda^2}{m_R^2}\right)\right) \\
\Rightarrow \delta m^{2^{(1)}} &= -\frac{\lambda_R}{32\pi^2}\left(\Lambda^2 - m_R^2\ln\left(\frac{\Lambda^2}{m_R^2}\right)\right).
\end{aligned}
$$

$\underline{\boldsymbol{\delta z_\phi^{(1)}}}$

$$
\begin{aligned}
-i\left.\frac{\mathrm{d}\,\Gamma_{\phi^2}\left(p^2\right)}{\mathrm{d}p^2}\right|_{p^2=m_R^2} &= z_\phi \\
&= 1+\delta z_\phi^{(1)} \\
&\overset{!}{=} 1 \text{ by condition 2.} \\
\Rightarrow \delta z_\phi^{(1)} &= 0.
\end{aligned}
$$

In order to calculate the coupling-constant renormalisation $\delta\lambda$, one must consider all graphs, i.e.



Two more renormalisation conditions are:

1. Infrared (IR) renormalisation: $\Gamma_{\Phi^4}\left(p_1,p_2,p_3,p_4\right)\big|_{p_i=0} = -i\lambda_R$

2. Symmetric renormalisation: $\Gamma_{\Phi^4}\left(p_1,p_2,p_3,p_4\right)\big|_{s=t=u=\frac{4}{3}m_R^2} = -i\lambda_R$.

One can now calculate the renormalisation factor $\delta\lambda$ for ϕ^4 :

$$\begin{aligned}\Gamma_{\phi^4} &= -i\left(\lambda_R+\delta\lambda^{(1)}\right)\left(1+2\delta z_\phi^{(1)}\right)+3\,\Gamma_s^{(1)}\Big|_{UV}\\ &= -i\lambda_R - i\delta\lambda^{(1)} - 2i\lambda\delta z_\phi^{(1)} + 3\,\Gamma_s^{(1)}\Big|_{UV}\\ &\stackrel{!}{=} -i\lambda_R \text{ by above condition.}\end{aligned}$$

Therefore, using $\delta z_\phi^{(1)} = 0$ and $\Gamma_s^{(1)}\Big|_{UV} = \frac{i\lambda_R^2}{32\pi^2}\ln\left(\frac{\Lambda^2}{m_R^2}\right)$;

$$\Rightarrow \delta\lambda^{(1)} = \frac{3\lambda_R^2}{32\pi^2}\ln\left(\frac{\Lambda^2}{m_R^2}\right).$$

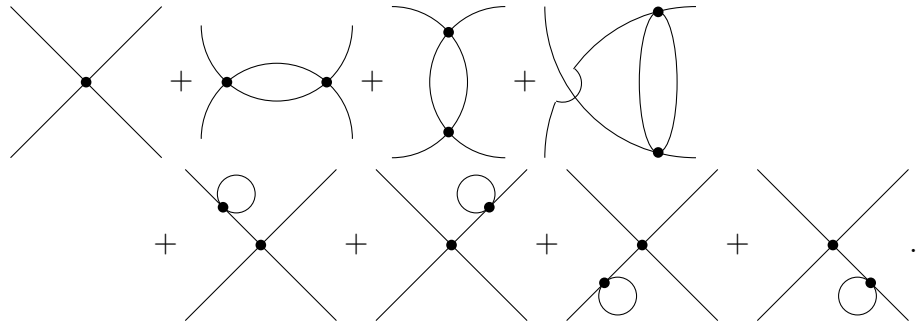
In summary, the renormalisation conditions are

1. Infrared (IR) renormalisation: $\Gamma_{\Phi^4}|_{p_i=0} = -i\lambda_R$
2. Polar propagator on-shell renormalisation: $\Gamma_{\phi^2}\big|_{p_i^2=m_R^2} = 0$
3. Symmetric renormalisation: $\Gamma_{\Phi^4}|_{s=t=u=\frac{4}{3}m_R^2} = -i\lambda_R$
4. Residual propagator on-shell renormalisation: $-i\,\dfrac{\mathrm{d}\,\Gamma_{\phi^2}\left(p^2\right)}{\mathrm{d}p^2}\Big|_{p^2=m_R^2} = 1$
5. Minimal subtraction scheme: $\Gamma_{\Phi^4}|_{UV} = 0$.

which result in the one loop renormalisation constants being:

$$\begin{aligned}\delta\Lambda_c^{(1)} &= -\frac{\Lambda^4}{64\pi^2};\\ z_\phi^{(1)} &= 1;\\ \delta z_\phi^{(1)} &= 0;\\ \delta m^{2^{(1)}} &= -\frac{\lambda_R}{32\pi^2}\left(\Lambda^2 - m_R^2\ln\left(\frac{\Lambda^2}{m_R^2}\right)\right);\\ \delta\lambda^{(1)} &= \frac{3\lambda_R^2}{32\pi^2}\ln\left(\frac{\Lambda^2}{m_R^2}\right).\end{aligned}$$

Previously only 3-level diagrams were covered, however 4-level (3-level + 1 loop) diagrams also exist in ϕ^4 theory, however they only contribute to mass renormalisation. The transition amplitude is then

$iT =$ 

The tree-level diagram is

$$\begin{aligned}\Gamma_{\phi^4} &= -i\lambda_0 \left(z_\phi^{\frac{1}{2}}\right)^4 \\ &= -i\left(\lambda_R + \delta\lambda^{(1)}\right)\left(1 + 2\underbrace{\delta z_\phi^{(1)}}_{=0}\right) \\ &= -i\left(\lambda_R + \delta\lambda^{(1)}\right),\end{aligned}$$

and the 3-level diagrams are

$$\Gamma_{\phi^4}^{(1)} = \Gamma_s^{(1)} + \Gamma_t^{(1)} + \Gamma_u^{(1)},$$

where $\Gamma_s^{(1)} = \tilde{\Gamma}\left[(p_1+p_2)^2\right] = \tilde{\Gamma}(s)$; $\Gamma_t^{(1)} = \tilde{\Gamma}\left[(p_1-k_1)^2\right] = \tilde{\Gamma}(t)$ and $\Gamma_u^{(1)} = \tilde{\Gamma}\left[(p_1-k_2)^2\right] = \tilde{\Gamma}(u)$. Using

$$\tilde{\Gamma}(p^2) = i\frac{\lambda_R^2}{32\pi^2}\ln\left(\frac{\Lambda^2}{m_R^2}\right) + \underbrace{\frac{i\lambda_R^2}{32\pi^2}\int_0^1 \frac{\mathrm{d}x\ \ (1-x)\,(1-2x)\,p^2}{m_R^2 - x\,(1-x)\,p^2 - i\epsilon}}_{:=\tilde{\Gamma}_{\text{fin}}\left(p^2\right)},$$

the transition amplitude is hence

$$\begin{aligned}iT &= -i\lambda_R - i\delta\lambda^{(1)} + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u) \\ &= -i\lambda_R + \tilde{\Gamma}_{\text{fin}}(s) + \tilde{\Gamma}_{\text{fin}}(t) + \tilde{\Gamma}_{\text{fin}}(u).\end{aligned}$$

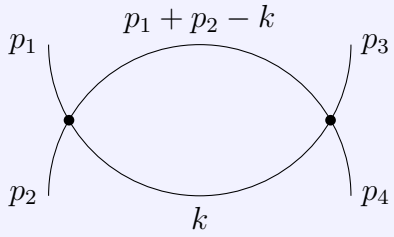
One should note that all UV infinities were removed by $\delta\lambda^{(1)}$ and $\delta m^{(1)}$. The transition amplitude iT only depends on λ_R and m_R, and it is therefore UV-finite, i.e. the cutoff Λ-singularity is removed.

Proof 5.2

The overall transition amplitude is given by

$$iT = -i\left(\lambda_R + \delta\lambda^{(1)}\right) + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u).$$

Consider the s diagram:



There are two intersections, thus

$$\tilde{\Gamma}(s) = \frac{(-i\lambda_0)^2}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}^4k}{(2\pi)^4}\left(\frac{i}{k^2 - m_0^2}\right)^2.$$

The mappings to Euclidean space have already been shown to be

$$k \mapsto i\,|k_E|\,; \qquad k^2 \mapsto -\,|k_E|^2\,; \qquad \mathrm{d}k \mapsto i\;\mathrm{d}k_E;$$

$$\begin{aligned}\mathrm{d}^4k \mapsto \mathrm{d}^4k_E &= i\,(2\pi)^2\,|k_E|^3\;\,\mathrm{d}\,|k_E| \\ &= i\,(2\pi)^2\cdot\frac{1}{2}\,|k_E|^2\;\,\mathrm{d}\,|k_E|^2 \\ &= i2\pi^2\,|k_E|^2\;\,\mathrm{d}\,|k_E|^2\,.\end{aligned}$$

Writing $\lambda_0 = \lambda_R + \delta\lambda^{(1)}$ and $m_0^2 = m_R^2 + \delta m^{2(1)}$ but ignoring perturbation terms, then

$$\begin{aligned}\tilde{\Gamma}(s) &= \frac{(-i\lambda_R)^2}{2}\int\frac{i2\pi^2\,|k_E|^2\;\,\mathrm{d}\,|k_E|^2}{(2\pi)^4}\frac{i^2}{\left(|k_E|^2 + m_R^2\right)^2} \\ &= \frac{i\lambda_R^2}{16\pi^2}\int\frac{x\;\mathrm{d}x}{\left(x + m_R^2\right)^2},\end{aligned}$$

where $x = |k_E|^2$.

5.3.2 d Dimensions

$d=6$	ϕ^3	Renormalisable	
	$\phi^{n>3}$	Non-renormalisable	
$d=4$	ϕ^3	Super-renormalisable	
	ϕ^4	Renormalisable	
	$\phi^{>4}$	Non-renormalisable	e.g. quantum gravity $\Rightarrow \mathcal{S} = \int \mathrm{d}^4x\ \sqrt{-g}\left(m_p^2 R + \Lambda_c\right)$
$d=3$	ϕ^4	Super-renormalisable	
	ϕ^6	Renormalisable	
	$\phi^{>6}$	Non-renormalisable	
$d=2$	ϕ^n	Renormalisable	

More succinctly:

d-**Dimensions**	**Renormalisable**	**Non-renormalisable**
6	ϕ^3	$\phi^{n>3}$
4	ϕ^4	$\phi^{n>4}$
3	ϕ^6	$\phi^{n>6}$
2	$\phi^{n\geq 2}$	-

5.4 Renormalisation of a Scalar Theory

Consider a "bare" Lagrangian $\mathcal{L}_0$ in ϕ^4-theory, i.e. one that contains potential infinities;

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu \phi_0\right)^2 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 + \underbrace{\Lambda_c^0}_{\substack{\text{cosmo-}\\ \text{logical}\\ \text{const.}}}$$

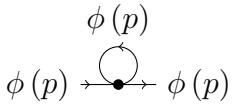
Since the ϕ^4-theory is renormalisable, all UV infinities of the 1PI loop graphs can also be absorbed into the bare Lagrangian parameters ϕ_0, m_0^2, λ_0 and

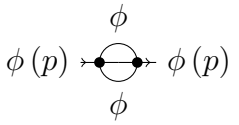
Λ_c^0. To this end, introduce the UV-finite (renormalised) parameters defined by:

$$\begin{aligned}\phi_0 &= z_\phi^{\frac{1}{2}}\phi_R;\\ m_0^2 &= z_{m^2}m_R^2;\\ \lambda_0 &= z_\lambda\lambda_R;\\ \Lambda_c^0 &= z_{\Lambda_c}\Lambda_c^R,\end{aligned}$$

where $z_x = 1+\delta z_x : x \in \left\{m^2, \lambda, \Lambda_c\right\}$ and $z_\phi^{\frac{1}{2}} = 1+\frac{1}{2}\delta z_\phi$ are renormalisable factors.

$N_\phi = 0$: $\Gamma^{(1)} =$  $= \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \sim \Lambda^4$

$N_\phi = 2$: $\Gamma^{(1)}_{\phi^2}(p) =$  (1 loop)

$\Gamma^{(2)}_{\phi^2}(p) =$  (2 loop)

both $\sim \underbrace{c_1\Lambda^2}_{\substack{\text{absorbed}\\ \text{into } m^2}} + \underbrace{c_2p^2\ln\left(\Lambda^2\right)}_{\substack{\text{absorbed}\\ \text{into } \phi}}$ + finite const.

$N_\phi = 4$: $\Gamma^{(1)}_{\phi^4} =$  $\sim \underbrace{\ln(\Lambda)}_{\substack{\text{absorbed}\\ \text{into } \lambda}}$ + finite const.

δz hence expands loopwise as

$$\delta z_x = \underbrace{\delta z_x^{(1)}}_{1\text{ loop}} + \underbrace{\delta z_x^{(2)}}_{2\text{ loop}} + \dots$$

Alternatively, one can decompose the bare terms into renormalisable and counter terms, i.e.

$$\left.\begin{aligned}\phi_0 &= \phi_R + \delta\phi\\ m_0^2 &= m_R^2 + \delta m^2\\ \lambda_0 &= \lambda_R + \delta\lambda\\ \Lambda_c^0 &= \Lambda_c^R + \delta\Lambda_c.\end{aligned}\right\} x_0 = x_R + \delta x$$

One can now cancel UV infinities of 1PI graphs by $\delta\phi$ and δx or δz_ϕ and δ_x. Firstly, calculate all Feynman graphs using the bare Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu \phi_0\right)^2 - \frac{1}{2} m_0^2 \phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 + \Lambda_c^0.$$

Then, define the renormalisation conditions to determine $\delta\phi$ and δx. For example, $\Lambda_c^R = 0$ to all orders in perturbation theory, i.e.

$$\Gamma_c = i\Lambda_c^0 + \overset{\phi_0}{\bigcirc} + \lambda_0 \overset{\phi_0}{\ominus} \lambda_0 + \ldots = i\Lambda_c^R = 0.$$

To one loop this is then

$$\begin{aligned} i\Lambda_c^0 + \frac{1}{2}\int_{-\Lambda}^{\Lambda} \frac{\mathrm{d}^4 k}{(2\pi)^4} &= i\Lambda_c^R + i\delta\Lambda_c + i\int_0^{\Lambda^2} \frac{k^2 \,\mathrm{d}\,|k|^2}{32\pi^2} \\ &= i\Lambda_c^R, \end{aligned}$$

and hence

$$\delta\Lambda_c = -\frac{\Lambda^4}{64\pi^2}.$$

Final Remarks

The UV-finite part is independent of the regularisation method. The same result would have been obtained for $\tilde{\Gamma}_{\text{fin}}\left(p^2\right)$ if dimensional regularisation have instead been used, whereby four dimensions are extended to $4 - 2\epsilon$. In such a case one gets

$$\begin{aligned} \tilde{\Gamma}\left(p^2\right) = \frac{i\lambda_R^2}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \ln\left(4\pi\right) - \ln\left(\frac{m_R^2}{\mu^2}\right)\right) \\ \underbrace{- \frac{i\lambda_R^2}{32\pi^2}\int_0^1 \ln\left(\frac{m_R^2 - x\left(1-x\right)p^2}{m_R^2}\right)\,\mathrm{d}x}_{= \frac{i\lambda_R^2}{32\pi^2}\int_0^1 \ln\left(\frac{(1-x)(1-2x)p^2}{m_R^2 - x(1-x)p^2}\right)\mathrm{d}x \;\equiv\; \tilde{\Gamma}_{\text{fin}}\left(p^2\right)}, \end{aligned}$$

where γ_E is the Euler constant. The correspondence between the covariant cutoff regularisation and dimensional regularisation is as follows:

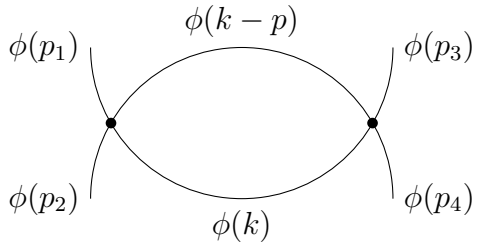
$$\frac{\lambda_R}{32\pi^2}\Lambda^2 \Leftrightarrow -\frac{\lambda_R m_R^2}{32\pi^2}\left(\frac{1}{\epsilon}+1-\gamma_E+\ln\left(4\pi\mu^2\right)\right)$$
$$\frac{\lambda_R}{32\pi^2}\ln\left(\frac{\Lambda^2}{m_R^2}\right) \Leftrightarrow \frac{\lambda_R}{32\pi^2}\left(\frac{1}{\epsilon}-\gamma_E+\ln\left(4\pi\mu^2\right)\right).$$

The renormalised quantities λ_R and m_R^2 depend on the conditions of renormalisation. However, physical observables, such as cross sections, are renormalisable scheme independent.

5.5 Mathematical Supplementary

5.5.1 Dimensional Regularisation

Reconsider the 1PI loop graph



where $p = p_1 + p_2 = p_3 + p_4$ and $p^2 = s$. The [...] for this is

$$\begin{aligned}\Gamma_s^{(1)} &= \frac{(-i\lambda)^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i^2}{(k^2-m^2)\left[(k-p)^2-m^2\right]}\\ &= \frac{\lambda^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(k^2-m^2)\left[(k-p)^2-m^2\right]}.\end{aligned}$$

Using Feynman parametrisation, i.e. $\frac{1}{AB} = \int_0^1 \frac{1}{\left[Ax+B(1-x)\right]^2}\,\mathrm{d}x$, then

$$\begin{aligned}
\Gamma_s^{(1)} &= \frac{\lambda^2}{2}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{\left[x\left(k^2+p^2-2k\cdot p-m^2\right)+(1+x)\left(k^2-m^2\right)\right]^2} \\
&= \frac{\lambda^2}{2}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4k}{(2\pi)^4} \underbrace{\frac{1}{\left[k^2-2x\left(k\cdot p\right)+xp^2-m^2+i\epsilon\right]^2}}_{=\underbrace{\left(k-xp\right)^2}_{:=\ell^2}+x(1-x)p^2-m^2+i\epsilon} \\
&= \frac{\lambda^2}{2}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4\ell}{(2\pi)^4} \frac{1}{\left[\ell^2-m^2+x\left(1-x\right)\left(p^2-i\epsilon\right)\right]^2}.
\end{aligned}$$

The integral to be solved, which will be denoted $I\left(\Delta\right)$, where $\Delta \coloneqq m^2 - x\left(1-x\right)p^2$, is now

$$I\left(\Delta\right) = \int \frac{\mathrm{d}^4\ell}{(2\pi)^4} \underbrace{\frac{1}{\left(\ell^2-\Delta^2-i\epsilon\right)^2}}_{=\ell_0^2-\left[\left(\ell^2+\Delta^2\right)^{\frac{1}{2}}-i\epsilon\right]^2},$$

where ℓ_0 is the temporal component of ℓ. To solve this, implement a Wick rotation, as shown in Figure 5.2.

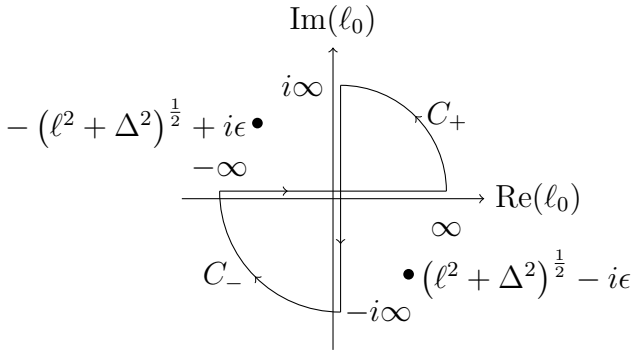


Figure 5.2

The contributions from the circular parts C_+ and C_- cancel each other, and

thus the integral becomes

$$
\begin{aligned}
I(\Delta) &= \int \frac{\mathrm{d}^3\ell}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{\mathrm{d}\ell_0}{2\pi} \frac{1}{\left[\ell_0^2 - \left((\ell^2+\Delta^2)^{\frac{1}{2}} - i\epsilon\right)\right]^2} \\
&= \int \frac{\mathrm{d}^3\ell}{(2\pi)^3} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\ell_0}{2\pi} \frac{1}{\left[\ell_0^2 - (\ell^2+\Delta^2) - i\epsilon\right]^2} \\
&= \int \frac{\mathrm{d}^3\ell}{(2\pi)^3} \left(i \int_{-\infty}^{\infty} \frac{\mathrm{d}\ell_4}{2\pi} \frac{1}{\left[\ell^2 + \ell_4^2 + \Delta^2 - i\epsilon\right]^2} \right),
\end{aligned}
$$

where $\ell_0 = i\ell_4$. The integral can then be seen to be an integral over 4D Euclidean space by setting $\ell_E = (\ell_1, \ell_2, \ell_3, \ell_4)$:

$$
I(\Delta) = i \int \frac{\mathrm{d}^4\ell_E}{(2\pi)^4} \frac{1}{\left(|\ell_E|^2 + \Delta^2 - i\epsilon\right)^2}.
$$

The dimensional regularisation method consists of calculating the integral $I(\Delta)$ in $n = 4 - 2\epsilon$ dimensions, instead of 4, where ϵ is vanishingly small.

5.6 Displacement Operator Formalism of Renormalisation to All Orders

5.7 Renormalisation Group Equation

5.8 Anomalous Magnetic Moment and the Lamb Shift